An explanation to the proof of (1.6) in Fictitious Play for Games with Identical Interests*

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Proof of (1.6) in the paper :

Note that for $s = t$ or $s = t+1$, $f(s) = (f^1(s), ..., f^n(s))$, where $f^i(s) \in \Delta^i$ is a vector, whose number of components equals the number of pure strategies of i. That is, $f^i(s) = (f^i(s)(y^i)_{y^i \in Y^i}$. U is formally defined over $f \in \Delta$. However, if you look for the definition of U at page 260 , you can see that it is actually defined for every $f = (f^1, f^2, ..., f^n)$, where f^i is a vector in R^{Y^i}. Hence, f^i can have coordinates that are above 1 or bellow zero. In the paper we assume without loss of generality that $\max_{f \in \Delta} U(f) \leq 2^{-n}$; this assumption is ok because we can always multiply U by a positive constant, and Δ is a compact set (i.e., U is bounded on Δ). However, to get (1.6) we actually need another assumption:

$$\max_{f \in \bar{\Delta}} U(f) \leq 1, \tag{1}$$

Where $\bar{\Delta}$ is the set of all $h = (h^1, h^2, ..., h^n)$ for which for every i, $h^i \in \Delta^i$ or $h^i = f^i - g^i$, where $f^i, g^i \in \Delta^i$. Since $\bar{\Delta}$ is also compact, there is no problem in this assumption.

Obviously, U is multi-linear (that is, it is linear in each coordinate separately). For example:

$$U(\alpha f^1 + \beta g^1, f^2, ..., f^n) = \alpha U(f^1, f^2, ..., f^n) + \beta U(g^1, f^2, ..., f^n). \tag{2}$$

Note therefore that $U(f^1 + g^1, f^2 + g^2, ...f^n + g^n)$ is the sum of 2^n factors. Formally:

$$U(f^1 + g^1, f^2 + g^2, ...f^n + g^n) = \sum_{\{S| \ S \in 2^N\}} U(h^1_S, ..., h^n_S), \tag{3}$$

where $h^i_S = f^i$ if $i \in S$, and $h^i_S = g^i$ if $i \notin S$, and 2^N denotes the class of all 2^n subsets of $N = \{1, ..., n\}$.

Recall now that by (1.2) in the paper,

$$f^i(t+1) = f^i(t) + \beta_t g^i(t), \tag{4}$$

where

$$g^i(t) = y^i(t+1) - f^i(t) \text{ and } \beta_t = 1/(t+1). \tag{5}$$

Hence,

$$U(f^1(t+1), ..., f^n(t+1)) = U(f^1(t) + \beta_t g^1(t), ..., f^n(t) + \beta_t g^n(t)). \tag{6}$$

Apply (3) to the right-hand-side of (6) to express it as the sum of 2^n factors. One of this factors is $U(f(t)) = U(f^1(t), ..., f^n(t))$;
There are exactly n factors in which $n-1$ coordinates are of the type $f^i(t)$ and one coordinate is of the type $\beta_t g^i(t)$. For example, if $n = 3$, the n factors are:
$U(\beta_t g^1(t), f^2(t), f^3(t))$, $U(f^1(t), \beta_t g^2(t), f^3(t))$, and $U(f^1(t), f^2(t), \beta_t g^3(t))$.

Use (5) to write each $g^i(t)$ as a difference of two expressions, and use the multi-linearity of U for each of these n factors. This will give you the first summand at the right-hand-side of (1.6) in the paper. In all other factors , $1/(t+1)$ is appearing at least twice. Using the multi-linearity, each of these factors has the form $1/(t+1)^k U^(h^1, h^2, ..., h^n)$, where $k \geq 2$, and each h^i is either in Δ^i or it has the form $h^i = s^i - t^i$, where each of s^i and t^i is in Δ^i. That is, $h = (h^1, h^2, ..., h^n) \in \bar{\Delta}$. As there are less than 2^n factors of this form, assumption (1) and the fact that $1/(t+1)^k \leq 1/(t+1)^2$ for every $k \geq 2$ yields (1.6) in the paper.