An explanation to the proof of (1.6) in Fictitious Play for Games with Identical Interests*

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Proof of (1.6) in the paper:

Note that for \( s = t \) or \( s = t + 1 \), \( f(s) = (f^1(s), \ldots, f^n(s)) \), where \( f^i(s) \in \Delta^i \) is a vector, whose number of components equals the number of pure strategies of \( i \). That is, \( f^i(s) = (f^i(s)(y'))_{y' \in \mathcal{Y}^i} \). \( U \) is formally defined over \( f \in \Delta \). However, if you look for the definition of \( U \) at page 260, you can see that it is actually defined for every \( f = (f^1, f^2, \ldots, f^n) \), where \( f^i \) is a vector in \( R^{\mathcal{Y}^i} \). Hence, \( f^i \) can have coordinates that are above 1 or below zero. In the paper we assume without loss of generality that \( \max_{f \in \Delta} U(f) \leq 2^{-n} \); this assumption is ok because we can always multiply \( U \) by a positive constant, and \( \Delta \) is a compact set (i.e., \( U \) is bounded on \( \Delta \)). However, to get (1.6) we actually need another assumption:

\[
\max_{f \in \Delta} U(f) \leq 1, \tag{1}
\]

Where \( \Delta \) is the set of all \( h = (h^1, h^2, \ldots, h^n) \) for which for every \( i \), \( h^i \in \Delta^i \) or \( h^i = f^i - g^i \), where \( f^i, g^i \in \Delta^i \). Since \( \Delta \) is also compact, there is no problem in this assumption.

Obviously, \( U \) is multi-linear (that is, it is linear in each coordinate separately). For example:

\[
U(\alpha f^1 + \beta g^1, f^2, \ldots, f^n) = \alpha U(f^1, f^2, \ldots, f^n) + \beta U(g^1, f^2, \ldots, f^n). \tag{2}
\]

Note therefore that \( U(f^1 + g^1, f^2 + g^2, \ldots, f^n + g^n) \) is the sum of \( 2^n \) factors. Formally:

\[
U(f^1 + g^1, f^2 + g^2, \ldots, f^n + g^n) = \sum_{\{S\} \in 2^{\mathcal{N}}} U(h^1_S, \ldots, h^n_S), \tag{3}
\]

where \( h^i_S = f^i \) if \( i \in S \), and \( h^i_S = g^i \) if \( i \notin S \), and \( 2^{\mathcal{N}} \) denotes the class of all \( 2^n \) subsets of \( \mathcal{N} = \{1, \ldots, n\} \).

Recall now that by (1.2) in the paper,

\[
f^i(t + 1) = f^i(t) + \beta_t g^i(t), \tag{4}
\]
where
\[ g^i(t) = y^i(t + 1) - f^i(t) \quad \text{and} \quad \beta_t = 1/(t + 1). \] (5)

Hence,
\[ U(f^1(t + 1), ..., f^n(t + 1)) = U(f^1(t) + \beta_t g^1(t), ..., f^n(t) + \beta_t g^n(t)). \] (6)

Apply (3) to the right-hand-side of (6) to express it as the sum of \( 2^n \) factors. One of this factors is \( U(f) = U(f^1, ..., f^n); \) There are exactly \( n \) factors in which \( n - 1 \) coordinates are of the type \( f^i(t) \) and one coordinate is of the type \( \beta_t g^i(t) \). For example, if \( n = 3 \), the \( n \) factors are:
\[ U(\beta_t g^1(t), f^2(t), f^3(t)), U(f^1(t), \beta_t g^2(t), f^3(t)), \text{and} U(f^1(t), f^2(t), \beta_t g^3(t)). \]

Use (5) to write each \( g^i(t) \) as a difference of two expressions, and use the multi-linearity of \( U \) for each of these \( n \) factors. This will give you the first summand at the right-hand-side of (1.6) in the paper. In all other factors, \( 1/(t + 1) \) is appearing at least twice. Using the multi-linearity, each of these factors has the form \( 1/(t + 1)^k U(h^1, h^2, ..., h^n), \) where \( k \geq 2 \), and each \( h^i \) is either in \( \Delta^i \) or it has the form \( h^i = s^i - t^i \), where each of \( s^i \) and \( t^i \) is in \( \Delta^i \). That is, \( h = (h^1, h^2, ..., h^n) \) \( \in \bar{\Delta} \). As there are less than \( 2^n \) factors of this form, assumption (1) and the fact that \( 1/(t + 1)^k \leq 1/(t + 1)^2 \) for every \( k \geq 2 \) yields (1.6) in the paper.