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MEASURE-BASED VALUES OF NONATOMIC GAMES*†

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A complete characterization of μ-symmetric linear continuous operators from $pNA(\mu)$ to FA is given. This is used to prove the existence of a unique μ-value on $pNA(\mu)$, and to deduce the general form of a μ-semivalue and the general form of a degenerate μ-value.

1. Introduction. Market games are games in characteristic function form that arise from pure exchange economies with transferable utility, or from production economies. Values of such games have been extensively analysed, in particular for "large" nonatomic games (e.g., see Aumann and Shapley [2, Chapter VI]).

Most of the known approaches—e.g. the asymptotic value [2] and Mertens' value [7]—take into account only the coalitional worth function. That is, the value of a market game depends only on the game derived from it; it thus suffices to know just the worth $v(S)$ of every coalition S, and no additional data from the economy are needed.

In order to extend the class of market games to which an asymptotic approach applies, the notion of a measure-based value has been introduced by Aumann and Kurz [1], and studied in Hart [6]. It takes into account, besides the coalitional worth function of the game, also the population measure μ of the underlying economy. This is done by considering only those partitions whose atoms have almost the same μ-measure.

This paper is devoted to *an axiomatic approach to measure-based values*. Aumann and Shapley [2] define a *value* as a linear operator, that associates to every game (in a certain subspace of games) a (bounded finitely-additive) measure, and satisfies the following axioms: efficiency, positivity and symmetry. When moreover a population measure μ is given on the space of players, one defines a μ-*value* in the same way, by replacing symmetry with μ-symmetry: only μ-preserving automorphisms of the space are considered, and adding one more axiom, the dummy axiom. Note that the dummy axiom appears in the Shapley value for finite games, and it is usually a consequence of the other axioms in the Aumann and Shapley nonatomic case; as we shall see in §2, this axiom is needed in order to eliminate some degenerate μ-symmetric operators, which depend essentially only on the population measure, and not on the game.

The first space of nonatomic games on which μ-values are studied is $pNA(\mu)$, the closed subspace of BV (games of bounded variation) spanned by all powers of probability measures which are absolutely continuous with respect to μ; this space contains all market games arising from differentiable markets with population measure μ. We will prove here that *there exists a unique* μ-*value on* $pNA(\mu)$. The difficult part is, of course, the uniqueness (the restriction of the unique value on pNA to $pNA(\mu)$ is easily seen to be a μ-value).

The Main Theorem in this paper provides a characterization of all μ-symmetric

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continuous linear operators from $pNA(\mu)$ into FA, the space of bounded finitely-additive measures. From this we deduce the general form of various classes of such operators, according to the different axioms they satisfy: in particular: the μ-value, degenerate μ-values (without the dummy axioms), μ-semivalues (without the efficiency axiom).

The paper is organized as follows: §2 is devoted to the precise definitions and the statements of our results; it also includes some general elementary properties of μ-values. In §3 we prove our main results, and the last §4 is devoted to some further remarks and open problems. In particular it will be explained how the uniqueness of the μ-value on $pNA(\mu)$ can be proved without using the Main Theorem.

2. Preliminaries and statement of results. Throughout this paper we shall use the terminology and notations established in [2], to which we add the following: (I, C) will be a fixed standard measurable space, G—the group of all automorphisms of I (that is, the group of all one-to-one measurable transformations from I onto itself having a measurable inverse). Every subset of I under discussion is supposed to be measurable. μ will denote a fixed nonatomic probability measure (i.e., $\mu \in NA^1$), $BV(\mu)$ will denote the space of all games $v \in BV$ for which:

$$\mu(S \triangle T) = 0 \Rightarrow v(S) = v(T), \qquad S, T \subseteq I$$

(where $\triangle$ denotes symmetric difference).

In addition, let: $FA(\mu) = FA \cap BV(\mu)$, $AC(\mu) = \{v \in BV : v \ll \mu\}$, $NA(\mu) = NA \cap BV(\mu)$ and $NA^1(\mu) = NA^1 \cap BV(\mu)$. $G(\mu)$ will be the subgroup of G consisting of all automorphisms of G which preserve the measure μ. A set $M \subseteq BV$ will be called μ-symmetric if for every $v \in M$ and every $\theta \in G(\mu)$, $\theta_* v \in M$.

We shall deal mainly with the space $pNA(\mu)$ which is defined to be the closed subspace of BV spanned by the powers of measures in $NA^1(\mu)$.

Finally, we remark that $BV(\mu)$ is a closed subspace of BV, that $pNA(\mu)$, $FA(\mu)$, $AC(\mu)$ and $NA(\mu)$ are all closed, μ-symmetric subspaces of $BV(\mu)$, and that (as can be easily seen from the proof of [2, Theorem J., Proposition 31.5 and Proposition 40.26]) $pNA(\mu)$ contains all market games arising from differentiable markets with population measure μ.

We now define the notion of a μ-value.

DEFINITION 2.1. Let Q be a μ-symmetric subspace of $BV(\mu)$. A μ-value on Q is a linear operator ψ, $\psi : Q \to FA$, which satisfies the following axioms:

(a-1) μ-Symmetry. For every $v \in Q$ and $\theta \in G(\mu)$, $\psi(\theta_* v) = \theta_* \psi v$.

(a-2) Positivity. For every monotone game $v \in Q$, $\psi v \geqslant 0$.

(a-3) Efficiency. For every $v \in Q$, $\psi v(I) = v(I)$.

(a-4) Dummy axiom. For every game v, if S is a null set of v, then $\psi v(S) = 0$. ∎

Recall that a set S is called a null set of v if S^c is a carrier of v, i.e. for every $T \subseteq I$, $v(T) = v(T \cap S^c)$. Note that in Definition 2.1 it is required that Q be a subspace of $BV(\mu)$ (rather than a subspace of BV). The reason for that is that for a "natural" measure μ on a population I, any two coalitions with symmetric-difference of μ-measure zero should be considered equal. Following this idea, it is natural to require also that $\psi Q \subseteq FA(\mu)$. However, this is not necessary since it follows from the other requirements (or, actually, from part of them):

LEMMA 2.2. *Let Q be a μ-symmetric subspace of $BV(\mu)$, and let $\psi : Q \to FA$ be a linear μ-symmetric operator. Then for every $v \in Q$, $\psi v \in FA(\mu)$.*

PROOF. Let $v \in Q$. All we have to show is that $\psi v(S) = 0$ for every null set S of μ. (Since ψv is an additive game, this would imply that $\psi v \in FA(\mu)$.) We divide the proof into two cases.

Case 1. *S is a finite set.* In this case, since ψv is additive, we have only to prove our claim for subsets consisting of one element.

Let, then, $x_0 \in I$. For every $x \in I$, $x \neq x_0$, we define $\theta \in G(\mu)$:

$$\theta x = x_0, \qquad \theta x_0 = x, \qquad \theta y = y, \qquad \forall y \neq x, x_0.$$

Since $v \in BV(\mu)$, it is obvious that $\theta_* v = v$, and since ψ is μ-symmetric we have also $\theta_* \psi v = \psi v$. In particular we get:

$$\psi v\{x\} = \psi v\{x_0\}. \tag{2.3}$$

Let $\{x_n : n \geqslant 1\}$ ($x_n \neq x_m$ for $n \neq m$) be any infinite countable subset of I. Since (2.3) holds for every $x \in I$, we conclude that for every $n \geqslant 1$, $\psi v\{x_1, x_2, \ldots, x_n\} = n\psi v\{x_0\}$.

Thus, if $\psi v\{x_0\} \neq 0$ then ψv is not bounded. This contradicts the assumption that ψv is in FA. Hence $\psi v\{x_0\} = 0$ as claimed.

Case 2. *S is an infinite subset of I (with* $\mu(S) = 0$). We partition S into two disjoint subsets, both with the same cardinality as $S : S = S_1 \cup S_2$.

By [2, Proposition 1.1] there exists a 1-1 measurable transformation τ from S_1 onto S_2 which is measurable in both directions. Define $\theta : I \to I$ as follows:

$$\theta x = \begin{cases} \tau x, & x \in S_1, \\ \tau^{-1} x, & x \in S_2, \\ x, & x \in (S_1 \cup S_2)^c. \end{cases}$$

Since S_1 and S_2 are null sets of μ, it follows that $\theta \in G(\mu)$, and since $v \in BV(\mu)$, $\theta_* v = v$. Thus $\theta_* \psi v = \psi v$ and, in particular, $\psi v(S_1) = \psi v(S_2)$. Similarly we partition S_2 into two disjoint subsets, both of the same cardinality as $S : S_2 = T_1 \cup T_2$. Considering the sets $T_1 \cup S_1$ and $T_2 \cup S_1$ and, using the same procedure as above, we obtain that $\psi v(T_1) = \psi v(S_1)$ and $\psi v(T_2) = \psi v(S_1)$. From this it follows that $\psi v(S_2) = 2\psi v(S_1)$, therefore $\psi v(S) = \psi v(S_i) = 0$. ∎

A linear operator from a μ-symmetric subspace of $BV(\mu)$ into FA, which satisfies axioms (a-1), (a-2), and (a-3) of Definition 2.1, but not necessarily the dummy axiom (a-4), will be called a *degenerate* μ*-value*.

Obviously, every μ-value is also a degenerated μ-value. The converse however is not true. For example, consider the operator $\tau : BV(\mu) \to FA$ defined by

$$\tau v = v(I)\,\mu. \tag{2.4}$$

As mentioned in the introduction, in most relevant cases the Aumann–Shapley value satisfies the dummy axiom. This was noted in [2, Note 4, p. 18], and we state it here explicitly, for the sake of completeness.

LEMMA 2.5. *Let Q be a symmetric subspace of BV, let* $\psi : Q \to FA$ *be a symmetric linear operator, and let* $v \in Q$ *be a game with an infinite number of null players (a player* $t \in I$ *is called a null player of v if* $\{t\}$ *is a null set of v). Then* $\psi v(S) = 0$ *for every null set S of v.*

From Lemma 2.5 we deduce also that if Q is a symmetric subspace of BV, which includes only games with an infinite number of null players, and if φ is a value on Q, then the restriction of φ to any μ-symmetric subspace of $Q \cap BV(\mu)$ is a μ-value. In particular, the restriction to $pNA(\mu)$ of the unique value on pNA is a μ-value. The question is: is that the only μ-value on $pNA(\mu)$? A positive answer is given in the following theorem.

THEOREM A. *There exists a unique* μ*-value* φ *on* $pNA(\mu)$; *it is the restriction to* $pNA(\mu)$ *of the unique value on* pNA. ∎

What about degenerate μ-values? We already have two different ones on $pNA(\mu)$, namely φ (the μ-value) and τ (defined in (2.4)). Then of course, any convex combination of the two is also a degenerate μ-value on $pNA(\mu)$. The question is: are these all the degenerate μ-values on $pNA(\mu)$? The answer is no; the following is a complete characterization.

THEOREM B. *Let ψ be a degenerated μ-value on $pNA(\mu)$. Then there exist constants $(A_n)_{n=1}^{\infty}$, $0 \leqslant A_n \leqslant 1$, s.t. for every $\lambda \in NA(\mu)$ and every $n \geqslant 1$,*

$$\psi\lambda^n = A_n\varphi\lambda^n + (1 - A_n)\tau\lambda^n. \quad \blacksquare$$

The next definition is that of a *μ-semivalue*. The notion of semivalue was introduced in [4], by replacing the efficiency axiom (a-1) with the following:

(a-5) Projection axiom. For every additive game γ, $\psi\gamma = \gamma$.

In [4] it was proved that for every semivalue ψ on pNA there exists some $f_0 \in L_\infty(0, 1)$, $f_0 \geqslant 0$ and $\int_0^1 f_0(x)\,dx = 1$, s.t. for every $v \in pNA$,

$$\psi v(S) = \int_0^1 \partial v(x, S) f_0(x)\,dx. \tag{2.6}$$

(dx stands for Lebesgue measure).

Following this approach, we define:

DEFINITION 2.7. Let Q be a μ-symmetric subspace of $BV(\mu)$. A μ-semivalue on Q is a linear operator ψ, $\psi : Q \to FA$ that satisfies axioms (a-1), (a-2), (a-4) and (a-5).

A degenerate μ-semivalue on Q is a linear operator $\psi : Q \to FA$ satisfying (a-1), (a-2) and (a-5).

THEOREM C. *Let ψ be an operator from $pNA(\mu)$ into FA. Then ψ is a degenerate μ-semivalue iff there exists some $f_0 \in L_\infty(0, 1)$, $f_0 \geqslant 0$ and $\int_0^1 f_0 = 1$, s.t.* (2.6) *holds for every $v \in pNA(\mu)$.* $\blacksquare$

In the course of proof of Theorem C we shall deduce that axiom (a-5), together with axioms (a-1) and (a-2), implies the dummy axiom (a-4), and thus Theorem A can be strengthened to get:

THEOREM D. *Let $\psi : pNA(\mu) \to FA$ be a linear operator satisfying axioms* (a-1), (a-2), (a-3) *and* (a-5). *Then ψ is the unique μ-value on $pNA(\mu)$.*

All the theorems we stated above will be consequences of the Main Theorem, in which we characterize μ-symmetric continuous linear operators on $pNA(\mu)$.

Before stating the main theorem we recall a few notations from [2] concerning directional derivatives of games in pNA. Every game $v \in pNA$ can be (uniquely) extended to a continuous game on $B(I, C)^1$—the set of all measurable functions on I which take values in the closed interval [0, 1]. We will denote the extension of v also by v. A game $v \in pNA$ is said to be differentiable at a point $0 < t < 1$, in the direction of a set $S \subseteq I$, if the limit

$$\lim_{s \to 0} \frac{v(t \cdot 1_I + s 1_S) - v(t 1_I)}{s}$$

exists. In this case we denote the limit by $\partial v(t, S)$. Further, v *is differentiable at t* if it is differentiable at t in the direction of S for all coalitions S.

By combining [2, Lemma 23.1] and [2, Lemma 24.1] we get:

LEMMA 2.8. *Let $v \in pNA$. Then for almost every $0 < t < 1$:*

(i) *v is differentiable at t.*

(ii) *$\partial v(t, \ )$ is a measure in NA.*

Furthermore, for every $S \subseteq I$, $\partial v(\ ,S)$ is Lebesgue integrable on [0, 1], *and*

$$\int_0^1 |\partial v(t,S)|\,dt \leq \|v\|. \quad ■$$

MAIN THEOREM. *For every μ-symmetric, continuous linear operator $\psi : pNA(\mu) \to FA$ there exists a unique pair (f_0, g_0) in $L_\infty \times L_\infty$ s.t. for every $v \in pNA(\mu)$ and every $S \subseteq I$ the following holds:*

$$\psi v(S) = \int_0^1 \partial v(x,S) f_0(x)\,dx + \left(\int_0^1 \partial v(x,I) g_0(x)\,dx\right)\mu(S). \tag{$*$}$$

The correspondence $(f_0, g_0) \leftrightarrow \psi$ defined in $(*)$ is a linear isomorphism between $L_\infty \times L_\infty$ and the space of μ-symmetric continuous linear operators from $pNA(\mu)$ into FA, and moreover:

(a) $\mathrm{Max}(\|f_0\|_\infty, \|g_0\|_\infty) \leq \|\psi\| \leq \|f_0\|_\infty + \|g_0\|_\infty$.

(b) ψ is positive iff $f_0 \geq 0$ and $g_0 \geq 0$.

(c) ψ satisfies the efficiency axiom iff $f_0 + g_0 = 1$.

(d) ψ satisfies the dummy axiom iff $g_0 = 0$.

(e) ψ satisfies the projection axiom iff $\int_0^1 g_0 = 0$ and $\int_0^1 f_0 = 1$. ■

To conclude this section we discuss the continuity properties of μ-values. Every linear operator on a finite-dimensional space, in particular the Shapley value, is continuous. This is not the situation when dealing with Aumann–Shapley values or μ-values. Here the continuity of the value is not implied by the axioms, except for special cases. One such special case is given by [3] (see also [2, Note 1, p. 34]) which ensures that every positive linear operator from a closed internal subspace of BV into FA is continuous. In view of this fact, it is of interest to note that:

LEMMA 2.9. *$pNA(\mu)$ is an internal space.*

The proof of Lemma 2.9 can be derived from [2, Lemma 7.18] and thus we shall omit it.

3. **Proofs.**

The proof of the Main Theorem is divided into two: we first show that for every (f_0, g_0) in $L_\infty \times L_\infty$, equation $(*)$ defines $\psi : pNA(\mu) \to FA$ as a μ-symmetric continuous linear operator, satisfying (a), (b), (c), (d) and (e). Then by a sequence of lemmas we show that for every μ-symmetric continuous linear operator $\psi : pNA(\mu) \to FA$ there exists a unique pair (f_0, g_0) in $L_\infty \times L_\infty$ s.t. $(*)$ holds. The linearity of the map $(f_0, g_0) \leftrightarrow \psi$ is self-evident. Before we start the proof of the Main Theorem we'll mention notations and results that will be needed later. We denote by $ac_0[0,1]$ or shortly, ac_0, the Banach space of all absolutely continuous functions on the interval [0, 1] which vanish at 0, with the variation norm. We shall denote by $P_0[0,1]$ the subspace of ac_0 consisting of all polynomials in ac_0.

It is known that the map $f \to f'$ is an isometry of ac_0 onto L_1. Thus we can regard L_∞ as the conjugate of ac_0. We denote the outcome of the action of $g \in L_\infty$ on $f \in ac_0$ by $g(f)$. That is, $g(f) = \int_0^1 f'(x) g(x)\,dx$.

Hence we get for every $g \in L_\infty$,

$$\|g\|_\infty = \sup\{\, g(f) : f \in ac_0 \text{ and } \|f\| \leq 1\}. \tag{3.1}$$

Finally:

LEMMA 3.2. *For every $\lambda \in NA^1(\mu)$ and every $f \in ac_0$, $f \circ \lambda \in pNA(\mu)$ and $\|f \circ \lambda\| = \|f\|$.* ■

The proof of the lemma is included implicitly in the proof of [2, Theorem C]. We now prove the first part of the Main Theorem.

LEMMA 3.3. *For every* $(f_0, g_0) \in L_\infty \times L_\infty$, *we define* $\psi : pNA(\mu) \to FA : \psi v(S) = \int_0^1 \partial v(x,S) f_0(x)\,dx + (\int_0^1 \partial v(x,I) g_0(x)\,dx)\mu(S)$, $v \in pNA(\mu)$, $S \subseteq I$.

Then ψ *is a* μ*-symmetric continuous linear operator, and properties* (a), (b), (c), (d) *and* (e) *of the Main Theorem hold.*

PROOF. The μ-symmetry and the linearity of ψ are self-evident. Proving (a) would prove also the continuity of ψ.

PROOF OF (a). We should prove first that for every $v \in pNA(\mu)$,

$$\|\psi v\| \leqslant (\|f_0\|_\infty + \|g_0\|_\infty)\|v\|. \tag{3.4}$$

We first show that (3.4) is true for monotone games. Let, then, $v \in pNA(\mu)^+$. We have:

$$\psi v(S) = \int_0^1 \partial v(x,S) f_0^+(x)\,dx - \int_0^1 \partial v(x,S) f_0^-(x)\,dx + \left(\int_0^1 \partial v(x,I) g_0(x)\,dx\right)\mu(S).$$

Each of the first two terms in the right-hand side of the above equality defines a positive measure. Hence,

$$\begin{aligned}\|\psi v\| &\leqslant \int_0^1 \partial v(x,I) f_0^+(x)\,dx + \int_0^1 \partial v(x,I) f_0^-(x)\,dx \\ &\quad + \left|\int_0^1 \partial v(x,I) g_0(x)\,dx\right| \\ &\leqslant \int_0^1 \partial v(x,I)(|f_0(x)| + |g_0(x)|)\,dx \\ &\leqslant \| |f_0| + |g_0| \|_\infty \int_0^1 \partial v(x,I)\,dx. \end{aligned} \tag{3.5}$$

It is known that for every $v \in pNA$, $\int_0^1 \partial v(x,I)\,dx = v(I)$, and that for a monotone game v, $v(I) = \|v\|$. Thus in (3.5) we actually have:

$$\|\psi v\| \leqslant \| |f_0| + |g_0| \|_\infty \|v\|, \tag{3.6}$$

and since $\| |f_0| + |g_0| \|_\infty \leqslant \|f_0\|_\infty + \|g_0\|_\infty$, this proves (3.4) for monotone games.

Now let v be any game in $pNA(\mu)$. Then for every $\delta > 0$ there exist (by the internality of $pNA(\mu)$) $u, w \in pNA(\mu)^+$ for which $v = u - w$ and $\|v\| \geqslant \|u\| + \|w\| - \delta$. Thus,

$$\begin{aligned}\|\psi v\| = \|\psi u - \psi w\| &\leqslant \|\psi u\| + \|\psi w\| \leqslant (\|f_0\|_\infty + \|g_0\|_\infty)(\|u\| + \|w\|) \\ &\leqslant (\|f_0\|_\infty + \|g_0\|_\infty)(\|v\| + \delta).\end{aligned}$$

Since the last inequality holds for every $\delta > 0$, (3.4) holds for any $v \in pNA(\mu)$. We now turn to prove the second inequality in (a). For every $T \subseteq I$, $0 < \mu(T) < 1$, denote: $\mu^T(S) = \mu(S \cap T)/\mu(T)$, $S \subseteq I$.

By Lemma 3.2, for every $T \subseteq I$, $0 < \mu(T) < 1$, and every $f \in ac_0$, $f \circ \mu^T \in pNA(\mu)$. Let us compute $\psi(f \circ \mu^T)$.

$$\psi(f \circ \mu^T) = \left(\int_0^1 f' f_0\right)\mu^T + \left(\int_0^1 f' g_0\right)\mu.$$

But for $t = \mu(T)$ we have $\mu = t\mu^T + (1-t)\mu^{T^c}$. Thus

$$\psi(f \circ \mu^T) = \left(\int_0^1 f' f_0 + t\int_0^1 f' g_0\right)\mu^T + \left((1-t)\int_0^1 f' g_0\right)\mu^{T^c}. \tag{3.7}$$

Hence for every $f \in ac_0$, $\|f\| \leqslant 1$, and for every $0 < t < 1$,

$$\|\psi\| \geqslant \|\psi(f \circ \mu^T)\| = \left|\int_0^1 f'f_0 + t\int_0^1 f'g_0\right| + (1-t)\left|\int_0^1 f'g_0\right|. \tag{3.8}$$

By letting t converge to 0 in (3.8) we get that for every $f \in ac_0$, $\|f\| \leqslant 1$,

$$\|\psi\| \geqslant \left|\int_0^1 f'f_0\right| + \left|\int_0^1 f'g_0\right|.$$

This, together with equation (3.1), yields $\|\psi\| \geqslant \operatorname{Max}(\|f_0\|_\infty, \|g_0\|_\infty)$ as required.

[REMARK. A. Neyman showed us that the norm of ψ is actually equal to $\| |f_0| + |g_0| \|_\infty$ (see (3.6)).]

PROOF OF (b). The "if" part is obvious. We shall prove the "only if" part. For every $T \subseteq I$ and $f \in ac_0^+$, $f \circ \mu^T \in pNA(\mu)^+$. Hence, by the positivity of ψ, $\psi(f \circ \mu^T) \geqslant 0$. Since the measures μ^T and μ^{T^c} are singular we get from equality (3.7) that

$$\int_0^1 f'g_0 \geqslant 0, \tag{3.9}$$

$$\int_0^1 f'f_0 + t\int_0^1 f'g_0 \geqslant 0. \tag{3.10}$$

Since (3.9) holds for every $f \in ac_0^+$ and (3.10) holds for every $t > 0$ and $f \in ac_0^+$, we get that $g_0 \geqslant 0$ and $f_0 \geqslant 0$.

PROOF OF (c). If $f_0 + g_0 = 1$ then for every $v \in pNA(\mu)$ we get:

$$\psi v(I) = \int_0^1 \partial v(x, I)(f_0(x) + g_0(x))\,dx = \int_0^1 \partial v(x, I)\,dx = v(I).$$

That is, ψ is efficient.

As for the converse—if ψ is efficient we get from the last equation that for every $v \in pNA(\mu)$,

$$\int_0^1 \partial v(\ , I)(f_0 + g_0) = v(I) = \int_0^1 \partial v(\ , I).$$

Hence for every $v \in pNA(\mu)$,

$$\int_0^1 \partial v(\ , I)(f_0 + g_0 - 1) = 0. \tag{3.11}$$

By substituting in (3.11) $v = f \circ \mu$, $f \in ac_0$, we get that for every $f \in ac_0$, $\int_0^1 f'(f_0 + g_0 - 1) = 0$. That is, $f_0 + g_0 = 1$.

PROOF OF (d). If $g_0 = 0$, then for every $v \in pNA(\mu)$,

$$\psi v(S) = \int_0^1 \partial v(\ , S) f_0. \tag{3.12}$$

The right-hand side of (3.12) defines a semivalue on pNA (recall (2.6) and [4]), and by Lemma 2.5 every semivalue on pNA satisfies the dummy axiom. Hence, ψ satisfies the dummy axiom.

And the converse: Let T be any set with $0 < t = \mu(T) < 1$. Since $\mu^T(T^c) = 0$ and $\mu^{T^c}(T^c) = 1$, we get by (3.7) that for every $f \in ac_0$,

$$\psi(f \circ \mu^T)(T^c) = (1-t)\int_0^1 f'g_0.$$

For every $f \in ac_0$, T^c is a null set of $f \circ \mu^T$ and thus, by the dummy axiom, $\psi(f \circ \mu^T)(T^c) = 0$. Hence, for every $f \in ac_0$, $\int_0^1 f' g_0 = 0$. That is, $g_0 = 0$.

PROOF OF (e). Assume that $\int_0^1 f_0 = 1$ and $\int_0^1 g_0 = 0$. For every additive game $\lambda \in pNA(\mu)$, $S \subseteq I$ and $x \in I$, $\partial\lambda(x, S) = \lambda(S)$. Thus by (*),

$$\psi\lambda(S) = \left(\int_0^1 f_0\right)\lambda(S) + \left(\int_0^1 g_0\right)\lambda(I)\,\mu(S) = \lambda(S).$$

That is, $\psi\lambda = \lambda$.

The converse: Assume that ψ satisfies the projection axiom. Let T be any set with $0 < t = \mu(T) < 1$. By (3.7),

$$\mu^T = \psi\mu^T = \left(\int_0^1 f_0 + t\int_0^1 g_0\right)\mu^T + \left((1-t)\int_0^1 g_0\right)\mu^{T^c}.$$

Hence $\int_0^1 g_0 = 0$ and $\int_0^1 f_0 = 1$ as claimed. ∎

And now for the proof of the second part of Theorem A. For every $T \subseteq I$, we denote $\mu_T(S) = \mu(S \cap T)$, $S \subseteq I$.

With this notation we state:

LEMMA 3.13. *Let $\psi : pNA(\mu) \to FA$ be a μ-symmetric, continuous linear operator, and $l \geqslant 1$. Let f be a real-valued function defined on $[0, 1]^l$ such that $f \circ (\mu_{T_1}, \mu_{T_2}, \ldots, \mu_{T_l}) \in pNA(\mu)$ for every sequence $T_1, \ldots, T_l$ of l pairwise disjoint subsets of I.*

Then for every $0 < t < 1/l$ there exists a unique sequence of $l+1$ real numbers, $\alpha_1(t), \alpha_2(t), \ldots, \alpha_l(t)$, $\beta(t)$, such that for every sequence of l pairwise disjoint subsets of I, $T_1, T_2, \ldots, T_l$, each of them with μ-measure t,

$$\psi(f \circ (\mu_{T_1}, \mu_{T_2}, \ldots, \mu_{T_l})) = \sum_{i=1}^{l} \alpha_i(t)\mu_{T_i} + \beta(t)\,\mu_{T^c},$$

where $T = \bigcap_{i=1}^{l} T_i$.

Furthermore, if f is symmetric in the variables i and j, $1 \leqslant i \neq j \leqslant l$, then for every $0 < t < 1/l$, $\alpha_i(t) = \alpha_j(t)$.

PROOF. Let ψ, l and f be as stated in the lemma. Let $0 < t < 1/l$ and let $T_1, T_2, \ldots, T_l$ be any sequence of l pairwise disjoint subsets of I, each of μ-measure t. Then the measures $\mu_{T_1}, \mu_{T_2}, \ldots, \mu_{T_l}, \mu_{T^c}$ are all linearly independent (since they are mutually singular). Thus the uniqueness part is clear. We shall therefore prove the existence of such constants.

Let $0 < t < 1/l$ and let $T_1, T_2, \ldots, T_l$ be a sequence of pairwise disjoint subsets of I, each with μ-measure t. For every two subsets of T_1 with equal μ-measure, S_1 and S_2, there exists by [5, p. 74] a 1-1 measurable transformation $\tau : T_1 \to T_1$, which preserves the measure μ, and for which:

$$\mu(\tau S_1 \triangle S_2) = 0. \tag{3.14}$$

Denote $\mu_i = \mu_{T_i}$ $(1 \leqslant i \leqslant l)$, $\bar{\mu} = \mu_{T^c}$, $v = f \circ (\mu_1, \mu_2, \ldots, \mu_l)$, and define $\theta \in G$ by:

$$\theta x = \begin{cases} \tau x, & x \in T_1, \\ x, & x \in T_1^c. \end{cases}$$

It is clear that $\theta \in G(\mu)$, and that $\theta_* v = v$. Thus the μ-symmetry of ψ yields:

$$\theta_* \psi v = \psi v. \tag{3.15}$$

By Lemma 2.2, $\psi v \in FA(\mu)$. Hence we get from (3.14) and (3.15) that

$$\psi v(S_1) = \psi v(S_2). \tag{3.16}$$

Since (3.16) is valid for every two subsets of T_1 with equal μ-measure, we deduce from [2, p. 39] that there exists a constant α_1 s.t. for every $S \subseteq T_1$, $\psi v(S) = \alpha_1 \mu_1(S)$.

Similarly we can show that there exist constants $\alpha_2, \ldots, \alpha_l, \beta$ such that for every $S \subseteq T_i$, $2 \leqslant i \leqslant l$, $\psi v(S) = \alpha_i \mu_i(S)$, and for every $S \subseteq T^c$, $\psi v(S) = \beta \bar{\mu}(S)$.

We thus have

$$\psi v = \sum_{i=1}^{l} \alpha_i \mu_i + \beta \bar{\mu}. \tag{3.17}$$

Let now $S_1, S_2, \ldots, S_l$ be another sequence of pairwise disjoint subsets of I each of μ-measure t. Denote $\mu_{S_i} = \lambda_i$ $(1 \leqslant i \leqslant l)$, $S = \bigcup_{i=1}^{l} S_i$, $\bar{\lambda} = \mu_{S^c}$, and $w = f \circ (\lambda_1, \lambda_2, \ldots, \lambda_l)$. By [2, Note 1, p. 40] there exists a $\theta \in G(\mu)$ s.t. for every $1 \leqslant i \leqslant l$, $\theta S_i = T_i$ and $\theta S^c = T^c$. Then clearly $\theta_* \mu_i = \lambda_i$, $1 \leqslant i \leqslant l$, and $\theta_* \bar{\mu} = \bar{\lambda}$. Thus $\theta_* v = w$, and by symmetry of ψ we deduce that

$$\psi w = \psi(\theta_* v) = \theta_* \psi v = \sum_{i=1}^{l} \alpha_i \theta_* \mu_i + \beta \theta_* \bar{\mu} = \sum_{i=1}^{l} \alpha_i \lambda_i + \beta \bar{\lambda}.$$

That is, the constants $\alpha_1, \alpha_2, \ldots, \alpha_l, \beta$ depend only on t and not on the particular sequence $T_1, T_2, \ldots, T_l$ we chose at first.

Finally assume that f is symmetric in the variables i and j. We shall show that $\alpha_i(t) = \alpha_j(t)$ for every $0 < t < 1/l$. Let, then, t be fixed, and let $T_1, T_2, \ldots, T_l$ be any sequence of pairwise disjoint subsets of I each with μ-measure t. By [2, Note 1, p. 40], there exists $\theta \in G(\mu)$ s.t. $\theta T_i = T_j$, $\theta T_j = T_i$, and $\theta x = x$ for every $x \in (T_i \cup T_j)^c$. Hence, $\theta_* \mu_i = \mu_j$, $\theta_* \mu_j = \mu_i$, and $\theta_* \mu_k = \mu_k$ for every $k \neq i, j$. This yields, by the symmetry of f in the variables i and j, that $\theta_* v = v$. Thus $\theta_* \psi v = \psi v$. Hence,

$$\begin{aligned} &\alpha_1(t)\mu_1 + \cdots + \alpha_i(t)\mu_i + \cdots + \alpha_j(t)\mu_j + \cdots + \alpha_l(t)\mu_l + \beta(t)\bar{\mu} \\ &\quad = \alpha_1(t)\mu_1 + \cdots + \alpha_i(t)\mu_j + \cdots + \alpha_j(t)\mu_i + \cdots + \alpha_l(t)\mu_l + \beta(t)\bar{\mu}. \end{aligned}$$

By the uniqueness part of the lemma, $\alpha_i(t) = \alpha_j(t)$. ∎

LEMMA 3.18. *Let $\psi : pNA(\mu) \to FA$ be a μ-symmetric, continuous linear operator and let $l \geqslant 1$. Then for every sequence of l positive integers $k_1, k_2, \ldots, k_l$, there exists a unique sequence of $l+1$ functions $\alpha_1(t), \alpha_2(t), \ldots, \alpha_l(t)$, $\beta(t)$, which are continuous on the interval $(0, 1/l)$ and have the following property:*

For every sequence of l pairwise disjoint subsets $T_1, T_2, \ldots, T_l$, each with μ-measure t,

$$\psi(\mu_1^{k_1}\mu_2^{k_2} \ldots \mu_l^{k_l}) = \sum_{i=1}^{l} \alpha_i(t)\mu_i + \beta(t)\bar{\mu},$$

where $\mu_i = \mu_{T_i}$, $T = \bigcup_{i=1}^{l} T_i$, $\bar{\mu} = \mu_{T^c}$.

Furthermore, if $k_i = k_j$ then $\alpha_i = \alpha_j$.

PROOF. Apply the previous lemma to the function $f(x_1, x_2, \ldots, x_l) = x_1^{k_1} x_2^{k_2} \cdots x_l^{k_l}$ to get the functions $\alpha_1(t), \ldots, \alpha_l(t)$, $\beta(t)$. We must show that these functions are continuous on the interval $(0, 1/l)$.

For every $0 < s < t < 1/l$ choose a sequence of l pairwise disjoint subsets each with μ-measure t, $T_1, T_2, \ldots, T_l$. For every $1 \leqslant i \leqslant l$ choose a subset $S_i \subseteq T_i$ with μ-measure s. Since $pNA(\mu)$ is a Banach algebra and ψ is continuous, $\|\psi(\mu_{T_1}^{k_1} \ldots \mu_{T_l}^{k_l}) - \psi(\mu_{S_1}^{k_1} \ldots \mu_{S_l}^{k_l})\|$ $(= A(s,t))$ tends to zero when s is constant and $t \to s+$, and when t is constant and $s \to t-$. Also,

$$\psi\left(\mu_{T_1}^{k_1}\cdots\mu_{T_l}^{k_l}\right)-\psi\left(\mu_{S_1}^{k_1}\cdots\mu_{S_l}^{k_l}\right)=\sum_{i=1}^{l}\alpha_i(t)\mu_{T_i}+\beta(t)\mu_{T^c}-\sum_{i=1}^{l}\alpha_i(s)\mu_{S_i}-\beta(s)\mu_{S^c}$$

$$=\sum_{i=1}^{l}(\alpha_i(t)-\alpha_i(s))\mu_{S_i}+(\beta(t)-\beta(s))\mu_{T^c}$$

$$+\sum_{i=1}^{l}(\alpha_i(t)-\beta(s))\mu_{T_i\setminus S_i}.$$

Thus for each $1 \leqslant i \leqslant l$,

$$|\alpha_i(t)-\alpha_i(s)|\cdot s \leqslant A(s,t),$$

and also

$$|\beta(t)-\beta(s)|(1-lt) \leqslant A(s,t).$$

Hence the continuity of $\alpha_i(t)$ and $\beta(t)$. ∎

LEMMA 3.19 (Main Lemma). *Let $\psi : pNA(\mu) \to FA$ be a μ-symmetric continuous linear operator. Then for every $N \geqslant 1$ there exist unique constants A_N and B_N s.t. for every $1 \leqslant l \leqslant N$, for every sequence of l positive integers $k_1, k_2, \ldots, k_l$ with $\sum_{i=1}^{l} k_i = N$, and for every sequence of l pairwise disjoint subsets of I, $T_1, T_2, \ldots, T_l$, all with equal μ-measure, $0 < \mu(T_1) < 1/l$, the following holds*:

$$\psi\left(\mu_{T_1}^{k_1}\mu_{T_2}^{k_2}\cdots\mu_{T_l}^{k_l}\right)=A_N\varphi\left(\mu_{T_1}^{k_1}\cdots\mu_{T_l}^{k_l}\right)+B_N\tau\left(\mu_{T_1}^{k_1}\cdots\mu_{T_l}^{k_l}\right),$$

where τ is the degenerate μ-value defined in 2.4 and φ is the value on pNA.

PROOF. Let $N \geqslant 1$. We first prove the claim for $l = N$. In this case we necessarily have $k_1 = k_2 = \cdots = k_l = 1$. Thus by Lemma 3.18 there exist continuous functions $\alpha(t)$ and $\beta(t)$, $0 < t < 1/l$, s.t. for every sequence of l pairwise disjoint sets T_1, $T_2, \ldots, T_l$ each with μ-measure t,

$$\psi(\mu_1\mu_2\cdots\mu_l)=\alpha(t)\sum_{i=1}^{l}\mu_i+\beta(t)\mu_{T^c}, \qquad \text{where} \tag{3.20}$$

$$\mu_i=\mu_{T_i}, \quad T=\bigcup_{i=1}^{l}T_i.$$

We now compute the functions $\alpha(t)$ and $\beta(t)$. Let $0 < t < 1/l$ and let T_1, $T_2, \ldots, T_l$ be a sequence of pairwise disjoint sets each with μ-measure t. For every $n \geqslant 2$ represent each of the sets T_i, $1 \leqslant i \leqslant l$, as a union of n pairwise disjoint sets, each with measure $t/n : T_i = \bigcup_{j=1}^{n} T_{ij}$. Denote $\mu_i = \mu_{T_i}$, $\mu_{i_j} = \mu_{T_{ij}}$. Under this notation we have for every $1 \leqslant i \leqslant l$,

$$\mu_i=\sum_{i=1}^{n}u_{i_j}, \quad \text{and also} \tag{3.21}$$

$$v=\prod_{i=1}^{l}\mu_i=\prod_{i=1}^{l}\left(\sum_{j=1}^{n}\mu_{i_j}\right)=\sum_{(j_1,\ldots,j_l)}\mu_{1_{j_1}}\mu_{2_{j_2}}\cdots\mu_{l_{j_l}}, \tag{3.22}$$

where the summation in the last term of (3.22) is over all n^l vectors $J = (j_1, \ldots, j_l)$ of l integers, where $1 \leqslant j_i \leqslant n$.

Denote $A_J = (\bigcup_{i=1}^{l} T_{i_{j_i}})^c$. It can be deduced from (3.22) that:

$$\psi v=\sum_{J}\left(\alpha(t/n)\sum_{i=1}^{l}\mu_{i_{j_i}}+\beta(t/n)\mu_{A_J}\right). \tag{3.23}$$

For every J, $A_J \supseteq T^c$. Hence

$$\mu_{A_J} = \mu_{T^c} + \mu_{A_J \setminus T^c}. \tag{3.24}$$

Substituting (3.24) in (3.23) and (3.21) in (3.20), and equating the coefficients of μ_{T^c}, yields

$$\beta(t) = \sum_J \beta(t/n) = n^l \beta(t/n). \tag{3.25}$$

Now define $u(t) = \beta(t)/t^l$, $0 < t < 1/l$. Since (3.25) holds for every $0 < t < 1/l$ and every $n \geqslant 1$, u is constant on the rational numbers in $(0, 1/l)$. Since u is continuous we deduce that u is a constant function. That is, there exists a constant B_N s.t. for every $0 < t < 1/l$,

$$\beta(t) = B_N t^N. \tag{3.26}$$

(Recall that $l = N$.)

In order to compute $\alpha(t)$, equate the coefficients of $\mu_{1_{j_1}}$ in equations (3.23) and (3.20) (after substituting (3.24) and (3.21) respectively). This yields

$$\alpha(t) = n^{N-1}\alpha(t/n) + (n^N - n^{N-1})\beta(t/n). \tag{3.27}$$

Define $w(t) = (\alpha(t) - B_N t^N)N/t^{N-1}$.

It is easily checked (using (3.27)) that $w(t/n) = w(t)$, $0 < t < 1/l$, $n \geqslant 1$. Thus w is constant in the interval $(0, 1/l)$. That is, there exists a constant A_N s.t. for every $0 < t < 1/l$,

$$\alpha(t) = A_N \cdot \frac{t^{N-1}}{N} + B_N t^N. \tag{3.28}$$

Now for every pairwise disjoint sets $T_1, T_2, \ldots T_l$, each with μ-measure t, and for any positive integers $k_1, k_2, \ldots, k_l$,

$$\varphi\left(\mu_{T_1}^{k_1} \cdots \mu_{T_l}^{k_l}\right) = \frac{t^{N-1}}{N} \sum_{i=1}^{l} k_i \mu_{T_i}, \tag{3.29}$$

$$\tau\left(\mu_{T_1}^{k_1} \cdots \mu_{T_l}^{k_l}\right) = t^N \mu. \tag{3.30}$$

In particular,

$$\begin{aligned} A_N \varphi(\mu_{T_1} \cdots \mu_{T_l}) + B_N \tau(\mu_{T_1} \cdots \mu_{T_l}) &= A_N \frac{t^{N-1}}{N} \sum_{i=1}^{l} \mu_{T_i} + B_N t^N \left(\sum_{i=1}^{l} \mu_{T_i} + \mu_{T^c} \right) \\ &= \left(A_N \frac{t^{N-1}}{N} + B_N t^N \right) \sum_{i=1}^{l} \mu_{T_i} + B_N t^N \mu_{T^c} \\ &= \alpha(t) \sum_{i=1}^{l} \mu_{T_i} + \beta(t)\mu_{T^c} = \psi(\mu_{T_1} \cdots \mu_{T_l}). \end{aligned}$$

We shall now prove by backwards induction on l, $1 \leqslant l \leqslant N$, that the result of the lemma is true, with the constants A_N and B_N found above. For $l = N$ we have just proved it. Let $l < N$ and assume our claim was already proved for all l', $l < l' \leqslant N$. Let $k_1, k_2, \ldots, k_l$ be positive integers for which $\sum_{i=1}^{l} k_i = N$. By Lemma 3.18 there exist continuous functions $\alpha_1(t), \alpha_2(t), \ldots, \alpha_l(t)$ and $\beta(t)$, where $0 < t < 1/l$, s.t. for every l pairwise disjoint subsets of I, $T_1, T_2, \ldots, T_l$, each with μ-measure t,

$$\psi\left(\mu_{T_1}^{k_1} \cdots \mu_{T_l}^{k_l}\right) = \sum_{i=1}^{l} \alpha_i(t)\mu_{T_i} + \beta(t)\mu_{T^c}. \tag{3.31}$$

Let then $0 < t < 1/l$ be fixed. Let $T_1, T_2, \ldots, T_l$ be any sequence of l pairwise disjoint sets with common measure t. For every $n \geqslant 2$ represent each of the sets T_i as a disjoint union of n sets, each with μ-measure t/n, $T_i = \bigcup_{j=1}^{n} T_{i_j}$.

Denote $\mu_i = \mu_{T_i}$, $\mu_{i_j} = \mu_{T_{i_j}}$. With these notations,

$$v = \prod_{i=1}^{l} \mu_i^{k_i} = \prod_{i=1}^{l} \left(\sum_{j=1}^{n} \mu_{i_j} \right)^{k_i} = \sum_{\mathbf{J}} \prod_{i=1}^{l} (\mu_{i_{j_1}i} \cdot \mu_{i_{j_2}i} \cdots \mu_{i_{j_{k_i}}i})$$

where the summation in the right-hand side is over all vectors $\mathbf{J}$, $\mathbf{J} = (J^1, J^2, \ldots, J^l)$, where $J^i = (j_1^i, j_2^i, \ldots, j_{k_i}^i)$ and $1 \leqslant j_p^i \leqslant n$ for $1 \leqslant p \leqslant k_i$.

Denote $v_{\mathbf{J}} = \prod_{i=1}^{l} (\mu_{i_{j_1}i} \cdot \mu_{i_{j_2}i} \cdots \mu_{i_{j_{k_i}}i})$.

For any $\mathbf{J} = (J^1, J^2, \ldots, J^l)$, if there exist $1 \leqslant i \leqslant l$ and $p_1 \neq p_2$ s.t. $j_{p_1}^i \neq j_{p_2}^i$, then $v_{\mathbf{J}}$ is a product of more than l measures. Thus, by induction hypothesis, $\psi v_{\mathbf{J}} = A_N \varphi v_{\mathbf{J}} + B_N \tau v_{\mathbf{J}}$. We shall call such $\mathbf{J}$ "a good $\mathbf{J}$." Any other $\mathbf{J}$ will be called "a bad $\mathbf{J}$."

We thus have:

$$v = \sum_{\substack{\text{good} \\ \mathbf{J}\text{'s}}} v_{\mathbf{J}} + \sum_{\substack{\text{bad} \\ \mathbf{J}\text{'s}}} v_{\mathbf{J}}.$$

Hence

$$\begin{aligned} \psi v &= \sum_{\substack{\text{good} \\ \mathbf{J}\text{'s}}} \psi v_{\mathbf{J}} + \sum_{\substack{\text{bad} \\ \mathbf{J}\text{'s}}} \psi v_{\mathbf{J}} \\ &= A_N \varphi v + B_N \tau v + \sum_{\substack{\text{bad} \\ \mathbf{J}\text{'s}}} (\psi v_{\mathbf{J}} - A_N \varphi v_{\mathbf{J}} - B_N \tau v_{\mathbf{J}}), \end{aligned}$$

which implies:

$$\begin{aligned} \|\psi v - A_N \varphi v - B_N \tau v\| &\leqslant \left\| \sum_{\substack{\text{bad} \\ \mathbf{J}\text{'s}}} (\psi v_{\mathbf{J}} - A_N \varphi v_{\mathbf{J}} - B_N \tau v_{\mathbf{J}}) \right\| \\ &\leqslant \sum_{\substack{\text{bad} \\ \mathbf{J}\text{'s}}} (\|\psi\| + |A_N|\,\|\varphi\| + |B_N|\,\|\tau\|)\|v_{\mathbf{J}}\| \\ &= (\|\psi\| + |A_N| + |B_N|) \sum_{\substack{\text{bad} \\ \mathbf{J}\text{'s}}} \|v_{\mathbf{J}}\|. \end{aligned}$$

For every $\mathbf{J}$ we have

$$\|v_{\mathbf{J}}\| = \prod_{i=1}^{l} (t/n)^{k_i} = (t/n)^N.$$

The number of bad $\mathbf{J}$'s is n^l. Thus from (3.32) we get that

$$\|\psi v - A_N \varphi v - B_N \tau v\| \leqslant n^{l-N} t^N (\|\psi\| + |A_N| + |B_N|). \tag{3.33}$$

Since $l < N$, and (3.33) holds for any $n > 1$, we get by letting $n \to \infty$ that $\psi v = A_N \varphi v + B_N \tau v$.

This completes the proof of the existence of constants A_N and B_N with the desired properties. Their uniqueness follows from equations (3.26) and (3.28), and from the uniqueness of the functions $\alpha(t)$ and $\beta(t)$ defined there. ∎

LEMMA 3.34. *Let $\psi : pNA(\mu) \to FA$ be a μ-symmetric continuous linear operator. Then there exist sequences of constants $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ s.t. for every $\lambda \in NA^+(\mu)$ and every $n \geqslant 1$, $\psi\lambda^n = A_n \varphi\lambda^n + B_n \tau\lambda^n$.*

PROOF. Let $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ be the constants from Lemma 3.19. For any finite sequence of pairwise disjoint sets, $T_1, T_2, \ldots, T_l$, with equal μ-measure, $0 < \mu(T_1) < 1/l$, and for any sequence of constants, $a_1, a_2, \ldots, a_l$, denote

$$\gamma = \sum_{i=1}^{l} a_i \mu_{T_i}. \tag{3.35}$$

We then have

$$\gamma^n = \left(\sum_{i=1}^{l} a_i \mu_{T_i} \right)^n = \sum_K a_1^{k_1} a_2^{k_2} \ldots a_l^{k_l} \mu_{T_1}^{k_1} \mu_{T_2}^{k_2} \ldots \mu_{T_l}^{k_l}$$

where summation is over all vectors $K = (k_1, k_2, \ldots, k_l)$ of l nonnegative integers s.t. $\sum_{i=1}^{n} k_i = n$. ψ, φ and τ are all linear, hence by Lemma 3.19, $\psi\gamma^n = A_n \varphi \gamma^n + B_n \tau \gamma^n$.

Since ψ, φ and τ are continuous, our task will be completed if we prove that the set of all measures of the form (3.35) is dense in $NA^+(\mu)$. It is an easy exercise in approximation of integrable functions by step functions and therefore we omit it.

END OF PROOF OF MAIN THEOREM. Let $\psi : pNA(\mu) \to FA$ be a μ-symmetric continuous linear operator. We shall prove that there exists a unique pair (f_0, g_0) in $L_\infty \times L_\infty$ s.t. (*) holds.

Let $(A_j)_{j=1}^{\infty}$ and $(B_j)_{j=1}^{\infty}$ be the constants of Lemma 3.19. For every polynomial P with $P(0) = 0$, $P(x) = \sum_{j=1}^{n} \alpha_j x^j$, we define:

$$A(P) = \sum_{j=1}^{n} \alpha_j A_j, \qquad B(P) = \sum_{j=1}^{n} \alpha_j B_j.$$

A and B are linear operators on $P_0[0, 1]$. We shall now prove that they are continuous.

For every $0 < t < 1$ choose a set T s.t. $\mu(T) = t$. By Lemma 3.34, for any polynomial $P(x) = \sum_{j=1}^{n} \alpha_j x^j$ in P_0 we have:

$$\begin{aligned} \psi(P \circ \mu^T) &= \sum_{j=1}^{n} \alpha_j \psi((\mu^T)^j) = \sum_{j=1}^{n} \alpha_j (A_j \mu^T + B_j \mu) \\ &= A(P)\mu^T + B(P)\mu \\ &= (A(P) + tB(P))\mu^T + (1-t)B(P)\mu^{T^c}. \end{aligned} \tag{3.36}$$

Since ψ is continuous,

$$\|\psi(P \circ \mu^T)\| \leq \|\psi\| \, \|P \circ \mu^T\| = \|\psi\| \, \|P\|.$$

Thus, for every $0 < t < 1$,

$$|A(P) + tB(P)| + (1-t)|B(P)| \leq \|\psi\| \, \|P\|. \tag{3.37}$$

By letting t tend to zero in (3.37) we get $|A(P)| + |B(P)| \leq \|\psi\| \, \|P\|$. That is, A and B are continuous operators. ($\|A\|, \|B\| \leq \|\psi\|$).

Since P_0 is dense in ac_0, A and B can be extended to continuous functionals on ac_0 (which we denote also by A and B).

Since L_∞ is the conjugate of ac_0, there exist functions $f_0 \in L_\infty$ and $g_0 \in L_\infty$ s.t. for every $f \in ac_0$ the following holds:

$$A(f) = \int_0^1 f' f_0, \qquad B(f) = \int_0^1 f' g_0. \tag{3.38}$$

We now show that (*) holds for these f_0 and g_0.

As shown in Lemma 3.3 the right-hand side of (*) defines a continuous linear operator on $pNA(\mu)$. Thus (since ψ is continuous and linear) it suffices to prove that

(*) holds for a set of games in $pNA(\mu)$ whose linear span is a dense subspace of $pNA(\mu)$. The powers of measures in $NA^1(\mu)$ form such a set. Hence the result follows from Lemma 3.34.

As for the uniqueness of the pair (f_0, g_0)—this follows directly from inequality (a). ∎

PROOF OF THEOREM A. Let ψ be a μ-value on $pNA(\mu)$. Since ψ is monotone, and $pNA(\mu)$ is closed and internal, ψ is continuous.

Thus by the Main Theorem there exist functions f_0 and g_0 in L_∞ such that (*) holds. Since ψ satisfies the dummy axiom, $g_0 = 0$. Since ψ is efficient, $f_0 + g_0 = 1$. Hence $f_0 = 1$. That is, for every $v \in pNA(\mu)$ and every $S \subseteq I$, $\psi v(S) = \int_0^1 \partial v(t, S)\,dt$.

Thus, by [2, Theorem H], ψ is the restriction of the value on pNA. ∎

PROOF OF THEOREM B. Let ψ be a degenerate μ-value on $pNA(\mu)$. Since ψ is monotone and $pNA(\mu)$ is closed and internal, ψ is continuous. Hence there exist functions f_0 and g_0 in L_∞ such that (*) holds.

For every $n \geqslant 1$ denote:

$$A_n = \int_0^1 nx^{n-1} f_0(x)\,dx, \qquad B_n = \int_0^1 nx^{n-1} g_0(x)\,dx.$$

Since ψ is monotone, $f_0 \geqslant 0$ and $g_0 \geqslant 0$. Since ψ is efficient, $f_0 + g_0 = 1$. Thus $A_n \geqslant 0$ and $B_n \geqslant 0$, $A_n + B_n = 1$ and, in particular, $0 \leqslant A_n \leqslant 1$.

For every $\lambda \in NA(\mu)$ and every $n \geqslant 1$ we have: $\partial\lambda^n(t, S) = nt^{n-1}\lambda(S)$. This, together with (*), implies:

$$\begin{aligned}\psi\lambda^n(S) &= \left(\int_0^1 nt^{n-1} f_0(t)\,dt\right)\lambda(S) + \left(\int_0^1 nt^{n-1} g_0(t)\,dt\right)\lambda(I)\,\mu(I)\\ &= A_n\lambda(S) + B_n\lambda(I)\,\mu(S).\end{aligned}$$

That is,

$$\psi\lambda^n = A_n\varphi\lambda^n + B_n\tau\lambda^n. \quad ∎$$

The proofs of Theorems C and D are analogous to the proofs of Theorems A and B, hence we shall omit them.

4. Remarks and open problems. I. Note that if in Lemma 3.19 ψ is taken to be a μ-value then, by the μ-value properties, it is easily seen that for every $N \geqslant 1$, $A_N = 1$ and $B_N = 0$. Thus in the proof of the lemma the computation of these constants is avoided (and the rest of the proof is simplified) and one gets a direct proof of the uniqueness of the μ-value on $pNA(\mu)$.

II. The interesting problems concerning μ-values are mainly questions of existence and uniqueness, some of which are presented below:

(i) Denote $Q_1 = pNA(\mu)$, $Q_2 = pNA \cap AC(\mu)$ and $Q_3 = pNA \cap BV(\mu)$. It is clear that Q_1, Q_2 and Q_3 are μ-symmetric subspaces of $BV(\mu)$, and that

$$Q_3 \supseteq Q_2 \supseteq Q_1. \tag{4.1}$$

The question is: are some of these inclusions actually equalities? If not, is the restriction of the value on pNA, the only μ-value on the space Q_i $(i = 2, 3)$?

(ii) Denote by $bv'NA(\mu)$ the closed subspace of BV spanned by all games of the form $f \circ \lambda$, $f \in bv'$, $\lambda \ll \mu$.

It is clear that $bv'NA(\mu)$ is a μ-symmetric subspace of $BV(\mu)$, and that the restriction of the unique value on $bv'NA$ to $bv'NA(\mu)$ is a μ-value. Is that the only μ-value on $bv'NA(\mu)$?

(iii) It is known (see [2, p. 19]) that there is no value on BV. Does there exist a μ-value on the whole of $BV(\mu)$?

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References

[1] Aumann, R. J. and Kurz, M. (1977). Power and Taxes in a Multi-Commodity Economy. *Israel J. Math.* 27 185–234.

[2] ——— and Shapley, L. S. (1974). *Values of Non-Atomic Games*. Princeton University Press, Princeton, N.J.

[3] Bakhtin, I. A., Krasnoselskii, M. A. and Stetzenko, V. Ya. (1962). On the Continuity of Linear Positive Operators. *Sibirsk. Mat. Z* **3** 156–160.

[4] Dubey, P., Neyman, A. and Weber, R. J. (1981). Value Theory without Efficiency. *Math. Oper. Res.* 6(1).

[5] Halmos, P. R. (1956). *Ergodic Theory*. Chelsea, New York.

[6] Hart, S. (1980). Measure-Based Values of Market Games. *Math. Oper. Res.* 5(2).

[7] Mertens, J. F. (1982). The Shapley Value in the Non-Differentiable Case. Core Discussion Paper 8240.

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