Values and Semivalues on Subspaces of Finite Games

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Abstract: It is proved that every value or semivalue on a linear symmetric subspace of finite games is the restriction to this subspace of a semivalue on the space of all finite games.

The theorem is proved for the space of all finite games on a fixed finite set of players, and for the space of all games with a finite support on an infinite set of players (the universe of players).

Introduction

Let \( G_N \) be the space of all games in characteristic form on \( N = \{1, 2, \ldots, n\} \), and let \( A_N \) be the subspace of \( G_N \) consisting of all additive games. A semivalue on \( G_N \) is (following [2]) an operator \( \psi: G_N \to A_N \) which is a linear symmetric positive projection on \( A_N \). A value on \( G_N \) is an efficient semivalue. A complete characterization of semivalues is given in [2]: An operator \( \psi: G_N \to A_N \) is a semivalue iff there exist constants \( (p_k)_{k=0}^{n-1} \) s.t. \( p_k \geq 0 \) for every \( 0 \leq k \leq n-1, \sum_{k=0}^{n-1} p_k \left( \begin{array}{c} n-1 \\ k \end{array} \right) = 1 \), and:

\[
\psi \sigma(i) = \sum_{k=0}^{n-1} p_k \sum_{S \subseteq N \setminus \{i\}} (\sigma(S \cup \{i\}) - \sigma(S)), \quad \sigma \in G_N, \quad 1 \leq i \leq n. \tag{*}
\]

Moreover, the correspondence \( \psi \mapsto (p_k)_{k=0}^{n-1} \) is linear and 1-1.

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As for values, there exists a unique value — the Shapley value, which is obtained by letting \( p_k = \frac{1}{n(n-1)} \) in (*). Given a single game \( v \in G \), one may look for a concept of solution for \( v \) which depends only on \( v \), or on games very similar to \( v \), or on games generated by \( v \), and not on all the games in \( G \). Such an approach was already taken in [3]. It is proved there that the potential axiom determines the Shapley value on the set consisting of \( v \) and all its subgames. In this paper we characterize values and semivalues on the smallest linear symmetric space which contains a given game or, more generally, on every linear symmetric subspace of \( G_N \).

The notions of values and semivalues on subspaces of \( G_N \) are yet to be defined. No problem arises in the case of a subspace \( Q \) containing \( A_N \). For such a subspace these notions may be defined exactly as for \( G_N \). But when \( Q \nsubseteq A_N \) the projection axiom is meaningless. Of course, it has an apparently natural generalization: \( \psi u = \mu \), \( \mu \in Q \cap A_N \). However, the concept of solution this generalized projection axiom leads to was found nonsatisfactory. Instead we suggest the Milnor axiom (see [4]): A player can’t get more than its maximal marginal contribution or less than its minimal marginal contribution. We define a semivalue on an arbitrary subspace of \( G_N \) as a linear symmetric Milnor operator. We shall prove that if \( Q \supseteq A_N \) then \( \psi : Q \to A_N \) is a linear Milnor operator if it is a linear positive projection. That is, our definition of a semivalue coincides in this case with the natural one.

We shall also prove that linear Milnor operators have the extension property: a linear Milnor operator on a subspace of \( G_N \) can be extended to a linear Milnor operator on \( G_N \). This result serves also in the proof of the main theorem of Chapter 1: Every semivalue on a subspace of \( G_N \) can be extended to a semivalue on \( G_N \). This is proved by a symmetrization process on the Milnor extenstions of the given semivalue. The main theorem may be explained from another point of view: Let \( \psi : G_N \to A_N \) be a semivalue. Let \( Q_\psi = \{ v \in G_N : \psi v(N) = v(N) \} \). Obviously, for every \( Q \subseteq Q_\psi \) \( \psi : Q \to A_N \) is a value. By the main theorem, every value on a subspace of \( G_N \) is obtained in this manner.

In Chapter 2 we interpret a finite game as a game with a finite support on an infinite set of players (the universe of players) and prove our main theorem in this context.

The definition of a Milnor operator, as well as the proof of the extension property, are quite easy (one just has to use Zorn’s Lemma instead of an induction process). The symmetrization process is, however, less trivial since the group of symmetries is an infinite one. We overcome this difficulty by using compactness arguments.
1 Values and Semivalues on Subspaces of $G_N$

Let $N = \{1, 2, \ldots, n\}$ be the players’ set which will be fixed throughout this chapter. Denote by $G_N$ the space of all games in characteristic form on $N$, and denote by $A_N$ the subspace of $G_N$ consisting of all additive games. We will identify $A_N$ with the $n$-dimensional Euclidean space $R^n$. That is, for every $u \in R^n$ and every $S \subseteq N$, $u(S) = \sum_{i \in S} u_i$. A game $v \in G_N$ is a monotonic game if $v(S) \leq v(T)$ for every $S \subseteq T \subseteq N$. For every $M \subseteq G_N$, $M^*$ will denote the set of all monotonic games in $M$. For every $u$ and $v$ in $G_N$ we will write $u \geq v$ or $v \leq u$ if $u - v$ is a monotonic game. Clearly, for every $\lambda$ and $\mu$ in $A_N$, $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for every $1 \leq i \leq n$). For every $v \in G_N$ we define $v^*$ and $v_*$ in $A_N$ as follows:

$$v^*(i) = \max_{S \subseteq N \setminus i} (v(S \cup \{i\}) - v(S))$$
$$v_*(i) = \min_{S \subseteq N \setminus i} (v(S \cup \{i\}) - v(S)), \quad 1 \leq i \leq n.$$

That is $v^*(i)$ and $v_*(i)$ are the maximum marginal contributions of player $i$, respectively. It is clear that $A_N$ with its usual order is a complete lattice. That is, every non-empty subset of $A_N$ which is order bounded from above has a least upper bound (l.u.b.) and every non-empty subset of $A_N$ which is order bounded from below has a greatest lower bound (g.l.b.). Indeed, if $\phi \neq M \subseteq A_N$ is bounded from above then $\lambda = \text{l.u.b.}(M)$ is the following additive game:

$$\lambda_i = \sup_{\mu \in M} \mu_i, \quad 1 \leq i \leq n.$$

By standard arguments using properties of max, min, l.u.b. and g.l.b., we have:

**Lemma 1:** Let $u$ and $v$ be in $G_N$ and $u \in A_N$ then

1. $v^* \geq v \geq v_*$.
2. $v^* = \text{g.l.b.} \{\lambda \in A_N : \lambda \geq v\}$.
3. $v_* = \text{l.u.b.} \{\lambda \in A_N : \lambda \leq v\}$. 
4. \((-v)^* = -v_\theta\) and \((-v)_\theta = -v^*

5. \((\alpha v)^* = \alpha v^*\) and \((\alpha v)_\theta = \alpha v_\theta\), \(\alpha > 0\).

6. \((v + u)^* \preceq v^* + u^*\) and \((v + u)_\theta \succeq v_\theta + u_\theta\).

7. \(\mu^* = \mu = \mu_\theta\)

8. \((v + \mu)^* = v^* + \mu\) and \((v + \mu)_\theta = v_\theta + \mu\). \(\square\)

The group of all symmetries of \(N\) will be denoted by \(H_N\). For every \(\theta\) in \(H_N\) we define a linear operator \(\theta_* : G_N \rightarrow G_N\) as follows:

\[(\theta_* v)(S) = v(\theta S), \quad S \subseteq N \text{ and } v \in G_N.\]

Clearly the following holds.

\[\theta_* v^* = (\theta_* v)^*\]

\[\theta_* v_\theta = \theta_* v_\theta.\] (1.2)

A set \(M \subseteq G_N\) is a symmetric set if for every \(\theta \in H_N\) and \(v \in M, \theta_* v \in M\). We will deal mainly with operators \(\psi : Q \rightarrow A_N\), where \(Q\) is a linear symmetric subspace of \(G_N\). Recall that \(\psi\) is a positive operator if \(\psi v \geq 0\) whenever \(v \geq 0\) and that if \(Q \supseteq A_N\) then \(\psi\) satisfies the projection axiom if \(\psi \mu = \mu\) for every \(\mu \in A_N\).

**Definition 1.3:** Let \(Q\) be a linear symmetric subspace of \(G_N\) s.t. \(Q \supseteq A_N\) and let \(\psi : Q \rightarrow A_N\). \(\psi\) is a semivalue if it is a linear, symmetric positive projection. \(\psi\) is a value if it is an efficient semivalue.

We now wish to extend the above definition to operators defined on subspaces of \(G_N\) not necessarily containing \(A_N\). For that matter we present:

**Minor axiom:**

\[v_\theta \preceq \psi v \preceq v^*.\]
We remark that for linear operators the Milnor axiom is equivalent to both the upper Milnor axiom (ψv ≤ v*) and the lower Milnor axiom (ψv ≥ v*).

Lemma 1.4: Let ψ : Q → AN be a linear operator, where Q is a linear subspace of GN s.t. Q ⊆ AN. Then ψ satisfies the Milnor axiom iff ψ is a positive projection.

Proof: Assume first that ψ satisfies the Milnor axiom. By Corollary 1.1 for every μ ∈ AN,

μ = μ* ≤ ψμ ≤ μ* = μ.

Therefore ψ is a projection.

ψ is also positive, for if v ∈ Q+, then since v ≥ 0 and 0 ∈ AN, Lemma 1.1 yields that v ≥ 0. Thus by the Milnor axiom, ψv ≥ v* ≥ 0. As for the converse:

Assume ψ is positive projection and let v ∈ Q. By Lemma 1.1 v ≥ v*. Therefore by the positivity and projection axioms,

ψv ≥ ψv* = v*.

Hence ψ satisfies the lower Milnor axiom and since ψ is linear, the result follows. □

By Lemma 1.4 the following is a generalization of Definition 1.3:

Definition 1.5: Let Q be a linear symmetric subspace of GN, and let ψ : Q → AN. Then:

ψ is a semivalue if it is linear, symmetric and satisfies the Milnor axiom.

ψ is a value if it is an efficient semivalue. □

In the following lemmas we deal with some extension properties of Milnor operators defined on subspaces of GN.

Lemma 1.6: Let Q be a linear subspace of GN, and let ψ : Q → AN be a linear operator which satisfies the Milnor axiom. Then, ψ can be extended to a linear positive projection ψ : Q + AN → AN.
Proof: Define \( \tilde{\psi} : Q + A_N \to A_N \) as follows:

\[
\tilde{\psi}(v + \mu) = \psi v + \mu,
\]

for \( v \in Q \) and \( \mu \in A_N \).

In order to prove that \( \tilde{\psi} \) is a well defined, linear positive projection it suffices to prove that \( \psi v + \mu \geq 0 \) whenever \( v \in Q, \mu \in A_N \) and \( v + \mu \geq 0 \). Indeed, if \( v + \mu \geq 0 \) then \( v \geq -\mu \). Thus by Lemma 1.1 \( v \geq -\mu \). Hence,

\[
\psi v \geq v \geq -\mu,
\]

which implies:

\[
\psi v + \mu \leq 0.
\]

\[\Box\]

Lemma 1.7: Let \( Q \supseteq A_N \) be a linear subspace of \( G_N \) and let \( \psi : Q \to A_N \) be a linear positive projection. Then \( \psi \) can be extended to a linear positive projection on \( G_N \).

Proof: Obviously it suffices to prove that for every \( u \notin Q \), \( \psi \) can be extended to a linear positive projection on \( Q + \langle u \rangle \), where \( \langle u \rangle \) is the linear space spanned by \( u \).

Let \( u \notin Q \). Define \( \tilde{\psi} : Q + \langle u \rangle \to A_N \) as follows:

\[
\tilde{\psi}(w + au) = \psi w + a\lambda,
\]

where \( w \in Q \), \( a \) is a real number and

\[
\lambda \equiv \max \{ \psi v : v \in Q \text{ and } v \leq u \}.
\]

Since \( u^* \in Q \) and \( u^* \leq u \), the set \( \{ \psi v : v \in Q \text{ and } v \leq u \} \) is nonempty (it contains \( \psi u^* = u^* \)), and ordered bounded from above by \( \psi u^* = u^* \). Thus \( \lambda \) is a well defined element of \( A_N \). (Recall that by Lemma 1.4 \( \psi \) satisfies the Milnor axiom.) Clearly \( \tilde{\psi} \) is a linear projection and we have to prove that it is positive. Assume \( v + a\lambda \geq 0 \). If \( a > 0 \)

then \( u \geq -\frac{1}{a} v \). Thus by the definition of \( \lambda \), \( \lambda \geq \psi \left( -\frac{1}{a} v \right) \); which implies: \( \psi v + a\lambda \geq 0 \).

If \( a < 0 \), then \( u \leq -\frac{1}{a} v \). Hence, for every \( w \) in \( Q \) s.t. \( w \leq u \), \( w \leq -\frac{1}{a} v \). By the positive of \( \psi \), \( \psi w \leq \psi \left( -\frac{1}{a} v \right) \) which yields that \( \lambda \leq \psi \left( -\frac{1}{a} v \right) \). Thus, \( \psi v + a\lambda \geq 0 \). \[\Box\]
Theorem 1.8 (Main Theorem of Chapter 1): Every semivalue on a linear symmetric subspace of $G_N$ is the restriction to this subspace of some semivalue on $G_N$.

Proof: Let $\psi : Q \to A_N$ be a simivalue, where $Q$ is a linear symmetric subspace of $G_N$. By Lemmas 1.6 and 1.7 $\psi$ can be extended to a linear positive projection $\tilde{\psi} : G_N \to A_N$. Define $\psi_1 : G_N \to A_N$ as follows:

$$\psi_1 = \frac{1}{n!} \sum_{a \in H_N} \theta^{-1} \tilde{\psi} \theta_a.$$

Clearly $\psi_1$ is a semivalue on $G_N$ and since $\tilde{\psi}$ is symmetric on $Q$, $\psi_1 = \tilde{\psi} = \psi$ on $Q$. \qed

Corollary 1.9: Let $\psi : Q \to A_N$ be a semivalue. Then there exist constants $(p_k)_{k=0}^{n-1}$ s.t. $p_k \geq 0$ for every $0 \leq k \leq n-1$, $\sum_{k=0}^{n-1} p_k \binom{n-k}{k} = 1$, and for every $v \in Q$ and every $i \in N$,

$$\psi(v(i)) = \sum_{k=0}^{n-1} p_k \sum_{S \subseteq \mathcal{N} \setminus i \atop |S| = k} (v(S \cup i) - v(S)).$$

Proof: Combine Theorem 1.8 with the result of [2] mentioned in the introduction. \qed

2 Games on the Universe of Players

Our terminology will be similar to the one given in [1] and in [2]. Let $U$ be an infinite set of players (the universe of players). A set function $v : 2^U \to R$ is monotonic if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq U$. A game on $U$ is a set function $v : 2^U \to R$ with $v(\emptyset) = 0$, which is the difference of two monotonic set functions. The linear space of all games is denoted by $BV(U)$. The space of all additive games in $BV(U)$ is denoted by $FA(U)$. For every $M \subseteq BV(U)$, $M^+$ will denote the set of all monotonic games in $M$, and $M_I$ will denote the set of all $v \in M^+$ for which $v(U) = 1$. For every $v \in BV(U)$ and $T \subseteq U$ we define $v_T(S) = v(S \cap T)$, $S \subseteq U$. A set $T \subseteq U$ is a support of a game $v$ if $v = v_T$. Let $M \subseteq BV(U)$. The set of all $v$ in $M$ for which $v = v_T$ is denoted by $M_T$. A game $v$ is a finite game if it has a finite support. The space of all finite games is denoted by $G$, and the space of all finite additive games is denoted by $A$. That is, $A = FA(U) \cap G$. We will identify $A$ and the set of all functions $\tau : U \to R$ with a finite support. More precisely:
for every such $\tau$ we associate $\mu \in A$ as follows:

$$
\mu(S) = \sum_{i \in S \cap N} \tau(i), \quad S \subseteq U,
$$

where $N$ is a finite support of $\tau$.

Let $H$ be the group of all symmetries of $U$. A set $T \subseteq U$ is a support of $\theta \in \hat{H}$ if $\theta i = i$ for every $i \in T$. Let $T \subseteq U$. The subgroup of $\hat{H}$ consisting of all $\theta$ which are supported in $T$ is denoted by $\hat{H}_T$. The subgroup of $\hat{H}$ consisting of all symmetries with a finite support is denoted by $H$. Let $J \subseteq \hat{H}$. A set $M \subseteq BV(U)$ is $J$-symmetric if $\theta \ast v \in M$ for every $v \in M$ and every $\theta \in J$. It is symmetric if it is $H$-symmetric. The notions of $J$-symmetry and symmetry are defined in an obvious manner also for operators defined on $J$-symmetric subspaces of $BV(U)$. Obviously every $H$-symmetric subset of $G$ is symmetric and every $H$-symmetric operator on a symmetric subspace of $G$ is symmetric. For every $\mu_1$ and $\mu_2$ in $A$ we define:

$$(\mu_1 \vee \mu_2)(i) = \max (\mu_1(i), \mu_2(i))$$

$$(\mu_1 \wedge \mu_2)(i) = \min (\mu_1(i), \mu_2(i)), \quad i \in U.$$ 

Clearly $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ belong to $A$ and they are the l.u.b. and g.l.b. of $\{\mu_1, \mu_2\}$ respectively. That is, $A$ is a linear lattice. For every $\nu$ and $\mu$ in $G$ we write $\nu \leq \mu$ or $\nu \geq \mu$ if $\nu - \mu$ is a monotonic game. Obviously, for $\mu$ and $\lambda$ in $A$, $\mu \geq \lambda$ if $\mu \vee \lambda \geq \lambda(i)$ for all $i$ in $U$. For every $\nu \in G$ define:

$$
\nu^*(i) = \sup_{S \subseteq U \setminus i} (\nu(S \cup i) - \nu(S))
$$

$$
\nu_\mu(i) = \inf_{S \subseteq U \setminus i} (\nu(S \cup i) - \nu(S)), \quad i \in U.
$$

Clearly $\nu^*$ and $\nu_\mu$ are in $A$ and for every support $N$ of $\nu$, $\nu^* = (\nu^*)_N$, $\nu_\mu = (\nu_\mu)_N$ and

$$
\nu^*_N(i) = \max_{S \subseteq N \setminus i} (\nu(S \cup i) - \nu(S))
$$

$$
\nu_\mu_N(i) = \min_{S \subseteq N \setminus i} (\nu(S \cup i) - \nu(S)), \quad i \in N.
$$
We now define Milnor operators, values and semivalues as in Chapter 1. Obviously Lemmas 1.1, 1.4, 1.6 and 1.7 continue to hold in this context. (In Lemma 1.7 one has to use Zorn’s Lemma instead of an induction process.)

A few more results are needed now. For every \( \mu \in A \) define \( \| \mu \| = \sum_{f \in U} |\mu(f)| \).

Obviously \( (A, \| \mu \|) \) is a normed space. For every \( \mu_1 \leq \mu_2 \in A \), the set \( B(\mu_1, \mu_2) = \{ \mu \in A : \mu_1 \leq \mu \leq \mu_2 \} \) is homeomorphic to the compact subset \( \{ x \in R^N : x_1 \leq \mu_2(x) \} \) of the euclidean space \( R^N \), where \( N \) is a common finite support of \( \mu_1 \) and \( \mu_2 \). Therefore \( B \) is a compact set. By Tychonoff’s theorem the topological product \( L = \prod_{\nu \in G} B(\nu, \nu^\ast) \) is a compact set too.

Let \( K \) be the set of all linear positive projections \( \psi : G \rightarrow A \) endowed with the topology of pointwise convergence. That is, for every directed set \( D \), for every net \( (\psi_\alpha)_{\alpha \in D} \) in \( K \) and for every \( \psi \in K \),

\[
\lim \psi_\alpha = \psi \text{ iff } \lim \| \psi_\alpha - \psi \| = 0 \quad \text{for every } \psi \in G.
\]

Since by Lemma 1.4 every \( \psi \in K \) satisfies the Milnor axiom, we can easily deduce that the transformation \( \psi \rightarrow (\psi \circ_\nu \psi)_{\nu \in G} \) is a homeomorphism of \( K \) onto a closed subset of \( L \). Therefore \( K \) is a compact topological space.

**Theorem 2.1** (Main Theorem of Chapter 2): Every semivalue on a linear symmetric subspace of \( G \) is the restriction to this subspace of some semivalue on \( G \).

**Proof:** Let \( \psi_1 : Q \rightarrow A \) be a semivalue, where \( Q \) is a linear symmetric subspace of \( G \). By Lemmas 1.6 and 1.7 \( \psi_1 \) can be extended to an element of \( K \). Let \( K \) be the compact subset of \( K \) consisting of all the extensions of \( \psi_1 \). For every finite subset \( N \) of \( U \) denote:

\[
\tilde{K}_N = \{ \psi \in \tilde{K} : \theta_{\ast}^{-1} \psi \theta_{\ast} = \psi \text{ for every } \theta \in H_N \}.
\]

(Recall that \( H_N = \{ \theta \in H : \theta(i) = i \ \forall \ i \in N^c \}. \)

Obviously \( \tilde{K}_N \) is a closed subset of \( \tilde{K} \). We now show that it is not empty. Indeed, choose \( \psi \in \tilde{K} \) and denote

\[
\tilde{\psi} = \frac{1}{n!} \sum_{\theta \in H_N} \theta_{\ast}^{-1} \psi \theta_{\ast},
\]

where \( n = |N| \).
Clearly $\bar{\psi} \in K_N$. Since for any finite number of finite subsets $N_1, N_2, \ldots, N_l$ of $U$, $\bigcap_{l=1}^{l} \tilde{K}_{N_l} = \tilde{K}_N$, where $N = \bigcap_{l=1}^{l} N_l$ we can use the finite intersection property of $\tilde{K}$ to deduce that $\bigcap_{N} \tilde{K}_N \neq \emptyset$. Obviously every $\psi_2$ in the intersection set is a semivalue on $G$ which extends $\bar{\psi}$.

References


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