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A Stochastic Competitive R&D Race Where "Winner Takes All"

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The paper considers a race among multiple firms that compete over the development of a product. The first firm to complete the development gains a reward, whereas the other firms gain nothing. Each firm decides how much to invest in developing the product, and the time it completes the development is a random variable that depends on the investment level. The paper provides a method for explicitly computing a unique Nash equilibrium, parametrically in the interest rate; for a given interest rate, the Nash equilibrium is determined in time that is linear in the number of firms. The structure of the solution yields insights about the behavior of the participants. Furthermore, an explicit expression for a unique globally optimal solution is obtained and compared to the unique Nash equilibrium.

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1. Introduction

We analyze a competitive market in which multiple firms compete over the development of a certain technology-intensive product or process. Each firm has to decide whether to enter the race and, if so, how much to invest. The competition is characterized as a *winner-takes-all* mechanism—the revenues are collected only by the firm that is the first to complete the project, whereas all other firms earn nothing. The winner collects the revenues upon the successful completion of the project. The durations of the development times of these projects are random, having exponential distributions with rates that are proportional to the levels of investment; these random duration times are assumed to be independent across firms. Revenue streams that result from winning the race are discounted and they are allowed to be random, while independent of completion times (provided that completion times and these random variables are independent).

The main contributions of this paper are the explicit representations of a unique Nash equilibrium and a globally optimal solution as functions of the system parameters including the interest rate. The computation of the unique Nash equilibrium involves ranking of the firms according to the product of two data parameters (expressing their technological and marketing efficiencies) and computing a *breakpoint* for each firm. A firm will be active (i.e., invest a positive amount) if and only if the interest rate is below its respective breakpoint. Hence, two consecutive breakpoints that capture the market's interest rate between them determine the set of active firms. Our analysis also provides closed-form expressions for the amounts that active firms will invest and their equilibrium utilities. This simple representation of the Nash equilibrium leads to a variety of conclusions regarding the sensitivity of the market activities, the utilities and the expenditures in the equilibrium to changes in firm-specific and environmental parameters. For example, we show that the equilibrium expenditure of a firm is bounded from above by 25% of its potential revenue from the project, independently of all other parameters of the problem. In addition, we show that the problem of maximizing the sum of the firms' utilities always has a solution in which at most one firm invests a positive amount and provides an explicit formula for the optimal investment amount and the optimal utility. In particular, our results demonstrate that the globally optimal solution exhibits concentration of effort whereas diversification is (usually) present in a Nash equilibrium. Surprisingly, we provide an example where a monopolistic firm in a globally optimal solution is not active in a Nash equilibrium (whereas other firms are). We also explore the system's efficiency, that is, the ratio of the utility of the unique Nash equilibrium and the globally optimal utility.

The problem of selecting R&D projects to invest in and the amount of investment in each of the selected projects is common in many markets (see, e.g., the examples given in Baye and Hoppe 2003). Several factors make this problem particularly difficult to handle. First, it is generally difficult to quantify and analyze the function that links the resources allocated to each project and its success probability (e.g., Shi 2003). Second, the duration of R&D projects may be long and the associated expenses may be substantial. For example, developing a new compound in the pharmaceutical industry typically requires 10–15 years and hundreds of millions of dollars (e.g., Halliday et al. 1997). The combination of long durations and large investments increases both the risk of failure and the corresponding consequences to the firm. Third, firms that engage in R&D projects face a risk of being beaten by competitors. In today's global environment, *time-to-market* is a crucial criterion. In many technology markets, a firm that is the first to introduce a new product gains a significant and lasting advantage over its competitors. A fourth complication that arises in real-world scenarios involves constraints on resources, which result in (endogenous) competition between the projects that a firm may develop simultaneously, in addition to the competition against its competitors.

Resource allocation problems in R&D settings have attracted significant attention in the operations research literature since the 1960s. Most of the early work in this area (e.g., Freeman 1960, Rosen and Souder 1965, Charnes and Stedry 1966, Kamien and Schwartz 1971, Lucas 1971) focused on developing *optimization* models capable of capturing the stochastic nature of the problem. Many of these articles assume a *static* framework in which the race consists of a single stage and therefore the competing firms cannot adjust their expenditures in view of their relative progress vis-*à*-vis their competitors (Blanning 1981). This line of work was followed by *dynamic* models in which some of the investment decisions are made only after observing realizations from prior stages of the projects (e.g., Deshmukh and Chikte 1977, Hopp 1987, Posner and Zuckerman 1990, Gottinger 1992). Following 20 years in which the optimization approach dominated the scene, the competitive approach penetrated the literature in the mid-1980s (e.g., Park 1987). The effort continued into the 1990s and onwards with contributions such as Spector and Zuckerman (1997), Gerchak and Parlar (1999), Gerchak and Kilgour (1999) and Golany and Rothblum (2008).

In parallel, competitive and cooperative models of R&D and patent races have been studied extensively in the economics literature (e.g., Scherer 1967; Loury 1979; Dasgupta and Stiglitz 1980; Lee and Wilde 1980; Reinganum 1981, 1982; Harris and Vickers 1985, 1987; Lippman and McCardle 1987; Milgrom and Roberts 1990; Szidarovszky and Okuguchi 1997; Doraszelski 2003; Vives 2005, and references therein). Most of the studies of R&D investments in competitive environments focused on existence results and qualitative analysis of the Nash equilibria for variants of the model we consider here (e.g., zero interest rate, continuous expenditures over time of a renewable resource and nonlinear cost of securing exponential coefficients). In contrast, as we already noted, the current paper presents explicit and efficiently computable expressions for the Nash equilibrium and the globally optimal solution and derives qualitative results using these expressions.

Our paper was inspired by a model developed by Gerchak and Parlar (1999). Whereas there are some similarities between our model and the one developed by Gerchak and Parlar, there are important differences—in their model firms invest in multiple R&D activities (whereas we assume a single R&D race); they include a budget constraint (whereas we account for the expenses in the objective function); they assume random completion times with a more general probabilistic structure than we do; they do not allow discounting (whereas we do); and they assume that the firms earn the same rewards (whereas we allow firm-specific rewards). As for analysis, Gerchak and Parlar solve their model (i.e., compute a Nash equilibrium) only for instances with two firms, two projects, and exponential distributions, and this is achieved by numerical methods. Also, neither Gerchak and Parlar nor other references listed above consider the global optimization problem and compare its solution with the Nash equilibrium.

The paper is organized as follows. Section 2 introduces the formal model and notation. In §3 we characterize a unique Nash equilibrium solution, provide an explicit formula for its computation and derive numerous implications of its structure. In §4 we address the problem of determining a globally optimal solution—we provide an explicit solution and compare it with the Nash equilibrium solution. Section 5 presents a numerical experiment that focusses on the robustness of the Nash equilibrium solution to errors in the assessment of problem parameters. Section 6 discusses the relations between our model and the well-known rent-seeking and Cournot models. It also presents some extensions of our results along with some directions for future research. Finally, an extensive (analytic) sensitivity analysis and a measure of the system efficiency are given in Appendices A and B, respectively.

2. The Formal Model

We consider an environment in which n firms compete in an R&D race. The decision variable of firm $i \in N \equiv \{1, \ldots, n\}$ is the amount x_i (in \$) that it allocates to funding the project. These amounts are assumed to be one-time installments that are invested at time $t = 0$ and cannot be retrieved regardless of the actual length of the project (e.g., purchase of equipment that can be used throughout the lifetime of the project, outsourcing contracts, etc.). The completion time of the project by firm i, denoted T^i, is a random variable that follows an exponential distribution whose rate depends linearly on the funding level x_i.[1] More specifically, when firm i allocates x_i to the project, the rate of the exponential distribution of T^i is $\alpha_i x_i$, where $\alpha_1, \ldots, \alpha_n$ are given positive constants (in $1/(\$ \times \text{time-unit})$). The T^is are assumed to be stochastically independent over firms (under all potential joint funding levels). The α_is can be interpreted as the *technological efficiency* of the firms in developing the product (or service).

The evolution of time is assumed to be continuous and a fixed interest rate $\rho > 0$ is applied. Consequently, a benefit received at time t is discounted by $e^{-\rho t}$. When firm i completes the project before all other firms, it earns a random revenue stream over time representing the benefits that it will receive thereon. The situation where benefits are gained by a firm only if it is the first to complete the project—is referred to as *winner-takes-all*. Let $S^i(t)$ denote the random instantaneous revenue rate that firm i will receive t units of time following the completion of the project (when it wins the race). Each of the random processes $S^i(\cdot)$ is assumed to be stochastically independent with $T^1, \ldots, T^n$ (but $S^1(\cdot), \ldots, S^n(\cdot)$ can be dependent). The discounted total value of benefits for firm i resulting from winning the race then equals $\int_0^\infty e^{-\rho(t+T^i)} S^i(t)\,dt = e^{-\rho T^i} \int_0^\infty e^{-\rho t} S^i(t)\,dt$. With E representing the expectation operator, it will be assumed that $R_i \equiv E[\int_0^\infty e^{-\rho t} S^i(t)\,dt]$ is finite and positive for every firm. The R_is can be interpreted as the *marketing efficiency* of the firms in converting the technology they seek to develop into income.

For $i \in N$, let $-i$ stand for the set $N\backslash\{i\}$ and let T^{-i} denote the random variable $\min_{k\in -i} T^k$. Standard results imply that each T^{-i} is exponentially distributed with rate $\sum_{k=1,\,k\neq i}^n \alpha_k x_k$ and is stochastically independent of T^i.

LEMMA 1. *Let $x_1, \ldots, x_n$ be the amounts that the firms allocate, respectively, to the project. Then the expected discounted revenue of firm i is $(R_i \alpha_i x_i)/(\sum_{k=1}^n \alpha_k x_k + \rho)$.*

PROOF. Given (fixed) $i \in N$, let $\lambda \equiv \alpha_i x_i$ and $\mu \equiv \sum_{k=1,\,k\neq i}^n \alpha_k x_k$. Also, let the indicator random variable corresponding to an event A be denoted by I_A. The expected discounted revenue of firm i is then given by:

$$E\left\{\int_0^\infty e^{-\rho(t+T^i)} S^i(t) I_{\{T^i<T^{-i}\}}\,dt\right\}$$
$$= E\left\{\int_0^\infty e^{-\rho t} S^i(t)\,dt\right\} E\left\{e^{-\rho T^i} I_{\{T^i<T^{-i}\}}\right\}$$
$$= R_i \int_0^\infty \left(\int_0^t e^{-\rho y} \lambda e^{-\lambda y}\,dy\right) \mu e^{-\mu t}\,dt = \frac{R_i \lambda}{\lambda+\mu+\rho}, \quad (1)$$

where the first equality follows from the independence assumption, the second one follows from the definition of R_i and from taking expectation with respect to the joint distribution of T^i and T^{-i}, and the last equality follows from standard integration. □

Let $\mathbb{R}_\oplus$ be the set of nonnegative real numbers. We shall use vector notation, e.g., $\alpha \equiv (\alpha_1, \ldots, \alpha_n)$ and $x \equiv (x_1, \ldots, x_n)$, in particular, the scalar product of α and x will be denoted $\alpha^T x$ $(= \sum_{k=1}^n \alpha_k x_k)$. Lemma 1 yields the following expressions for the utility (i.e., net profit) of the firms as functions of the joint funding levels:

$$U_i(x) = \frac{R_i \alpha_i x_i}{\alpha^T x + \rho} - x_i \quad \text{for } i \in N. \quad (2)$$

Henceforth, the analysis will focus on a system where firms' utility functions are given by (2) (without consideration of how these expressions are derived). In particular, the data for the problem consists of the α_is, the R_is and ρ; the interpretation of the α_is and the R_is as technological and marketing efficiencies, respectively, will prevail.

For $x \in \mathbb{R}_\oplus^n$ and $i \in N$, let x_{-i} stand for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$; we then write $x = (x_i, x_{-i})$. A *Nash equilibrium* is a vector x^* that provides a simultaneous solution for n optimization problems where the action of each firm maximizes its utility when the actions of the other firms are determined by the respective coordinates of x^*, i.e.,

$$U_i(x^*) = \max_{x_i \geqslant 0} U_i(x_i, x_{-i}^*) \quad \text{for } i \in N. \quad (3)$$

3. Explicit Representation of a Unique Nash Equilibrium

This section develops an explicit expression for a unique Nash equilibrium, parameterized by the interest rate ρ.

Let X^* be the set of all Nash equilibria; then, $x^* \in X^*$ if and only if it simultaneously solves the n optimization problems given by (3), i.e.,

$$x_i^* \in \arg\max_{x_i \geqslant 0} U_i(x_i, x_{-i}^*) \quad \text{for } i \in N. \quad (4)$$

To explore the first-order conditions for (4), we observe that

$$\frac{\partial U_i}{\partial x_i}(x) = \frac{R_i \alpha_i (\alpha^T x - \alpha_i x_i + \rho)}{(\alpha^T x + \rho)^2} - 1 \quad \text{and} \quad (5)$$

$$\frac{\partial^2 U_i}{\partial x_i^2}(x) = \frac{-2R_i \alpha_i^2 (\alpha^T x - \alpha_i x_i + \rho)}{(\alpha^T x + \rho)^3}. \quad (6)$$

As (6) implies that each U_i is concave in x_i, it follows that the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for each of the optimization problems in (4) (the single constraint clearly satisfies the constraint qualification condition). The standard formulation of these conditions for a vector x^* asserts that for each $i \in N$ there exists $\beta_i \in \mathbb{R}$ such that: (i) $\beta_i \geqslant 0$, (ii) $x_i^* \geqslant 0$, (iii) $(\partial U_i/\partial x_i)(x^*) + \beta_i = 0$, and (iv) $x_i^* \beta_i = 0$. The next lemma reformulates these conditions using the explicit expression of the U_is.

LEMMA 2. *A vector x^* is a Nash equilibrium if and only if for all $i \in N$:*

$$\frac{R_i \alpha_i (\alpha^T x^* - \alpha_i x_i^* + \rho)}{(\alpha^T x^* + \rho)^2} \leqslant 1, \quad (7)$$

$$x_i^* \geqslant 0, \quad \textit{and} \quad (8)$$

$$[x_i^* > 0] \;\Rightarrow\; \left[\frac{R_i \alpha_i (\alpha^T x^* - \alpha_i x_i^* + \rho)}{(\alpha^T x^* + \rho)^2} = 1\right]. \quad (9)$$

Proof. The lemma follows immediately from the KKT conditions with

$$\beta_i^* = -\frac{\partial U_i}{\partial x_i}(x^*) = 1 - \frac{R_i\alpha_i(\alpha^T x^* - \alpha_i x_i^* + \rho)}{(\alpha^T x^* + \rho)^2}$$

(the last equality by (5)). □

Lemma 2 leads to the following explicit representation of Nash equilibria.

Theorem 1. *Let* $\mathcal{Q}$ *be the set of pairs* (F, I) *where*:
(a) F *is the unique positive solution of the equation*

$$\left(\sum_{i\in I}\frac{1}{R_i\alpha_i}\right)z^2 - (|I|-1)z - \rho = 0 \tag{10}$$

(*with* $|I|$ *denoting the cardinality of* I), *and*
(b) $I = \{i \in N\colon R_i\alpha_i > F\}$.
Then the correspondence that maps $(F, I) \in \mathcal{Q}$ *into* $x^* \in \mathbb{R}^n$ *with*

$$x_i^* = \begin{cases} \dfrac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} & \textit{if } i \in I, \\ 0 & \textit{otherwise} \end{cases} \tag{11}$$

is one to one from $\mathcal{Q}$ *onto* X^*. *Furthermore, the inverse correspondence that maps* $x^* \in X^*$ *into* $(F, I) \in \mathcal{Q}$ *has*

$$F = \alpha^T x^* + \rho \quad \textit{and} \tag{12}$$

$$I = \{i \in N\colon x_i^* > 0\}. \tag{13}$$

Proof. If $I \neq \varnothing$, then (10) is a quadratic equation in which z^0 has a negative coefficient and z^2 has a positive coefficient, so it has a unique positive root. Alternatively, if $I = \varnothing$, then (10) is the linear equation $z - \rho = 0$, which has a unique (positive) root.

Consider a pair $(F, I) \in \mathcal{Q}$ and let x^* be defined by (11). From (11) and (a)

$$\begin{aligned}\alpha^T x^* + \rho &= \sum_{k\in I}\frac{\alpha_k F(R_k\alpha_k - F)}{R_k\alpha_k^2} + \rho \\ &= |I|F - F^2\sum_{k\in I}\frac{1}{R_k\alpha_k} + \rho = F.\end{aligned} \tag{14}$$

To show that x^* is a Nash equilibrium, we verify (7)–(9). From (11), (b), and the positivity of F, it follows that $x^* \geqslant 0$ and that $x_i^* > 0$ if and only if $i \in I$; the latter and (11) imply that if $x_i^* > 0$, then $x_i^* = (F(R_i\alpha_i - F))/(R_i\alpha_i^2)$ that (using (14)) is equivalent to $(R_i\alpha_i(\alpha^T x^* - \alpha_i x_i^* + \rho))/((\alpha^T x^* + \rho)^2) = 1$. This proves (8) and (9). Next, for $i \in I$, (7) is trivial (by (11)). For $i \in N\backslash I$, we have from (b), (11), and (14) that $R_i\alpha_i \leqslant F$ and $(R_i\alpha_i(\alpha^T x^* - \alpha_i x_i^* + \rho))/((\alpha^T x^* + \rho)^2) = (R_i\alpha_i F)/F^2 \leqslant 1$, completing the verification of (7). Next, to show that the correspondence defined by (11) is one to one, we show that (F, I) satisfies (12)–(13). Indeed, (14) verifies (12). Also, $i \notin I$ implies $x_i^* = 0$ and $i \in I$ implies (by (b)) $R_i\alpha_i > F$ and (by (11) and the positivity of F) $x_i^* > 0$.

Now consider a Nash equilibrium x^* and let (F, I) be as in (12)–(13). To verify $(F, I) \in \mathcal{Q}$, we use the fact that x^* must satisfy (7)–(9) (by Lemma 2). From (9) and (12), for $i \in I$,

$$\begin{aligned}[x_i^* > 0] \;\Rightarrow\; & \left[\frac{F^2}{R_i\alpha_i(F - \alpha_i x_i^*)} = 1\right] \\ \Leftrightarrow\; & \left[\frac{F^2}{R_i\alpha_i} = F - \alpha_i x_i^*\right];\end{aligned} \tag{15}$$

as (13) assures that $\sum_{i\in I}\alpha_i x_i^* = \sum_{i\in N}\alpha_i x_i^* = \alpha^T x^*$, we conclude (using (12))

$$\begin{aligned}\left(\sum_{i\in I}\frac{1}{R_i\alpha_i}\right)F^2 &= |I|F - \sum_{i\in I}\alpha_i x_i^* = |I|F - \sum_{i\in N}\alpha_i x_i^* \\ &= (|I|-1)F + \rho,\end{aligned}$$

verifying (a). Next, from (13) and (9)

$$\begin{aligned}[i \in I] \;\Rightarrow\; [x_i^* > 0] \;\Rightarrow\; & \left[\alpha_i x_i^* = F - \frac{F^2}{R_i\alpha_i}\right] \\ \Leftrightarrow\; & \left[x_i^* = \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2}\right],\end{aligned} \tag{16}$$

implying that $R_i\alpha_i > F$ for $i \in I$. Also, (13) implies that for $i \in N\backslash I$, $x_i^* = 0$; as (7) and (12) imply that $R_i\alpha_i(F - \alpha_i x_i^*) \leqslant F^2$, it follows that $0 = \alpha_i x_i^* \geqslant R_i\alpha_i F - F^2 = F(R_i\alpha_i - F)$, assuring that $R_i\alpha_i \leqslant F$. This completes the proof of (b). Finally, to verify that the correspondence defined by (11) is onto and that (12)–(13) define its inverse, we show that x^* is the image of (F, I) under (11). Indeed, (12) shows that $x_i^* = 0$ for $i \in N\backslash I$ and (16) shows that $x_i^* = (F(R_i\alpha_i - F))/(R_i\alpha_i^2)$ for $i \in I$. □

Henceforth, we refer to conditions (a) and (b) without explicit reference to Theorem 1.

Theorem 1 implies that the search for Nash equilibria reduces to a search for pairs $(F, I) \in \mathcal{Q}$; the corresponding Nash equilibrium can then be constructed by (11). Evidently, given a subset I, condition (a) uniquely determines F, and given F, Condition (b) uniquely determines I. Therefore, $(F, I) \in \mathcal{Q}$ means that F and I are determined from each other by conditions (a) and (b), respectively. Theorem 1 also yields the following relationship between the firms' equilibrium expenditures.

Corollary 1. *Suppose* x^* *is a Nash equilibrium. If* $x_i^* > 0$ *and* $R_j\alpha_j > R_i\alpha_i$, *then* $x_j^* > 0$ *and* $\alpha_j x_j^* > \alpha_i x_i^*$.

Proof. Let $(I, F) \in \mathcal{Q}$ correspond to x^* through the correspondence in Theorem 1. By (13), $i \in I$ and because $R_j\alpha_j > R_i\alpha_i$, condition (b) implies that $j \in I$. Then, by (11) and $R_j\alpha_j > R_i\alpha_i$, we have

$$\begin{aligned}\alpha_i x_i^* - \alpha_j x_j^* &= \frac{F(R_i\alpha_i - F)}{R_i\alpha_i} - \frac{F(R_j\alpha_j - F)}{R_j\alpha_j} \\ &= F^2\left[\frac{1}{R_j\alpha_j} - \frac{1}{R_i\alpha_i}\right] < 0. \quad \square\end{aligned}$$

Corollary 1 establishes the following restriction on subsets I that can be paired with a scalar F to satisfy condition (b):

$$[(i \in I) \text{ and } (R_j\alpha_j > R_i\alpha_i)] \quad \Rightarrow \quad [j \in I]. \tag{17}$$

To simplify notation, we relabel the firms so that

$$R_1\alpha_1 \geqslant R_2\alpha_2 \geqslant \cdots \geqslant R_n\alpha_n; \tag{18}$$

further, to exclude degenerate situations, we assume that the $R_i\alpha_i$s are distinct, ensuring that the inequalities of (18) hold strictly. It then follows from (17) that a subset I that can be paired with a scalar F to satisfy condition (b) must have the form $I_m \equiv \{1, \ldots, m\}$ for some $m \in \{0, 1, \ldots, n\}$ (with $I_0 \equiv \varnothing$); Condition (b) with $I = I_m$ is then equivalent to the condition $R_{m+1}\alpha_{m+1} \leqslant F < R_m\alpha_m$.

For $m \in \{0, 1, \ldots, n\}$, let

$$\gamma_m \equiv \sum_{j \in I_m} \frac{1}{R_j\alpha_j} \tag{19}$$

(with $\gamma_0 \equiv 0$). Also, with $\mathbb{R}_+$ as the set of positive real numbers, let F_m: $\mathbb{R}_+ \to \mathbb{R}$ where for $\rho > 0$, $F_0(\rho) = \rho$ and

$$F_m(\rho) = \frac{(m-1) + \sqrt{(m-1)^2 + 4\gamma_m\rho}}{2\gamma_m} \quad \text{for } m \geqslant 1. \tag{20}$$

Evidently, $F_m(\rho)$ is the unique scalar F that is determined from $I = I_m$ by condition (a). Thus, the pair $(I_m, F_m(\rho))$ is in $\mathscr{Q}$ (and corresponds to a Nash equilibrium) if and only if $F_m(\rho)$ determines I_m by condition (b), that is, $R_{m+1}\alpha_{m+1} \leqslant F_m(\rho) < R_m\alpha_m$; our goal is to find values m that satisfy these two inequalities.

We next define firm-specific constants whose comparison with the interest rate is critical to our analysis. Specifically, let

$$\rho_m \equiv \gamma_m(R_m\alpha_m)^2 - (m-1)R_m\alpha_m \quad \text{for } m = 1, \ldots, n, \tag{21}$$

$$\rho_0 \equiv \infty \quad \text{and} \quad \rho_{n+1} \equiv -\infty. \tag{22}$$

In particular, $\rho_1 = R_1\alpha_1$ and $\rho_2 = ((\alpha_2R_2)^2)/(R_1\alpha_1) < R_1\alpha_1 = \rho_1$ (assuring that ρ_1 and ρ_2 are always positive). The computation of the ρ_ms for $m \geqslant 3$ is readily available from (21), as is demonstrated by the next example.

Example 1. Let $n = 4$, $\alpha_j = 1$ for $j \in \{1, 2, 3, 4\}$, $R_1 = 10$, $R_2 = 5$, $R_3 = 2$, and $R_4 = 1$. From (21), we have $\rho_1 = (1/10)10^2 = 10$, $\rho_2 = (1/10 + 1/5)5^2 - 5 = 2.5$, $\rho_3 = (1/10 + 1/5 + 1/2)2^2 - 2 \times 2 = -0.8$, and $\rho_4 = (1/10 + 1/5 + 1/2 + 1/1)1^2 - 3 \times 1 = -1.2$.

For $m \in \{1, \ldots, n\}$, use (20) to extend $F_m(\rho)$ to $\rho \leqslant 0$ satisfying $(m-1)^2 + 4\gamma_m\rho \geqslant 0$, that is, to $\rho \in [-((m-1)^2/(4\gamma_m)), 0]$. Also, extend $F_0(\rho)$ to $\rho \leqslant 0$ by $F_0(\rho) = \rho$. We observe that the $F_m(\cdot)$s are strictly increasing on their extended domains. The next result demonstrates, among other facts, that the ranking of the ρ_js in Example 1 is not a coincidence.

Lemma 3. (i) *For $m = 1, \ldots, n$, ρ_m is in the extended domain of $F_m(\cdot)$ and $F_{m-1}(\cdot)$ and $F_m(\rho_m) = R_m\alpha_m = F_{m-1}(\rho_m)$.*

(ii) $-\infty = \rho_{n+1} < \rho_n < \cdots < \rho_1 < \rho_0 = \infty$.

Proof. Part (i). For $m \geqslant 1$,

$$\begin{aligned}(m-1)^2 &+ 4\gamma_m\rho_m \\ &= (m-1)^2 + 4\gamma_m\big[\gamma_m(R_m\alpha_m)^2 - (m-1)R_m\alpha_m\big] \\ &= \big[(m-1) - 2\gamma_mR_m\alpha_m\big]^2 \geqslant 0,\end{aligned}$$

and therefore ρ_m is in the extended domain of $F_m(\cdot)$. Also, for $m \geqslant 2$,

$$\begin{aligned}(m-2)^2 &+ 4\gamma_{m-1}\rho_m \\ &= (m-2)^2 + 4\gamma_{m-1} \\ &\quad \cdot \big[\gamma_{m-1}(R_m\alpha_m)^2 + R_m\alpha_m - (m-1)R_m\alpha_m\big] \\ &= \big[(m-2) - 2\gamma_{m-1}R_m\alpha_m\big]^2 \geqslant 0,\end{aligned}$$

i.e., ρ_m is in the extended domain of $F_{m-1}(\cdot)$. Finally, the fact that ρ_1 is in the extended domain of $F_0(\cdot)$ is trivial.

By the representation of $F_m(\cdot)$ in (20), $F_m(\rho) = R_m\alpha_m$ if and only if

$$\sqrt{(m-1)^2 + 4\gamma_m\rho} = 2\gamma_mR_m\alpha_m - (m-1);$$

squaring both sides and isolating ρ implies that $F_m(\rho) = R_m\alpha_m$ if and only if $\rho = \gamma_m(R_m\alpha_m)^2 - (m-1)R_m\alpha_m = \rho_m$. Also, for $m \in \{2, \ldots, n\}$, $F_{m-1}(\rho) = R_m\alpha_m$ if and only if

$$\begin{aligned}\rho &= \gamma_{m-1}(R_m\alpha_m)^2 - (m-2)R_m\alpha_m \\ &= \gamma_m(R_m\alpha_m)^2 - (m-1)R_m\alpha_m = \rho_m.\end{aligned}$$

Finally, because $\rho_1 = R_1\alpha_1$, $F_0(\rho_1) = \rho_1 = R_1\alpha_1$.

Part (ii). For $1 \leqslant m \leqslant n-1$, Part (i) implies that $F_m(\rho_m) = R_m\alpha_m > R_{m+1}\alpha_{m+1} = F_m(\rho_{m+1})$. As each $F_m(\cdot)$ is strictly increasing, we conclude that $\rho_m > \rho_{m+1}$. □

Example 1 (Continued). In Example 1, $\gamma_1 = 0.1$, $\gamma_2 = 0.3$, $\gamma_3 = 0.8$, and $\gamma_4 = 1.8$. Then by (20), $F_0(\rho) = \rho$, $F_1(\rho) = \sqrt{10\rho}$, $F_2(\rho) = (1 + \sqrt{1 + 1.2\rho})/0.6$, $F_3(\rho) = (2 + \sqrt{4 + 3.2\rho})/1.6$, and $F_4(\rho) = (3 + \sqrt{9 + 7.2\rho})/3.6$. These increasing (concave) functions are depicted in Figure 1 where ρ_js are marked on the x-axis and $R_j\alpha_j$s are marked on the y-axis.

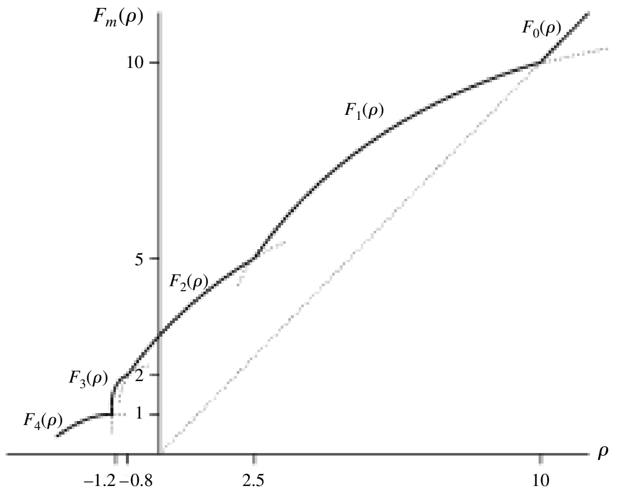
Theorem 2. *Suppose $0 \leqslant m \leqslant n$ is the unique integer with $\rho_{m+1} \leqslant \rho < \rho_m$. Then*:

(i) $\mathscr{Q} = \{(F_m(\rho), I_m)\}$.

(ii) *The vector x^* defined by* (11) *with $(F, I) = (F_m(\rho), I_m)$ is a unique Nash equilibrium in the domain of randomized strategies and*

$$U_i(x^*) = \begin{cases} R_i\left(1 - \dfrac{F}{R_i\alpha_i}\right)^2 & \text{if } i \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Figure 1. $F_m(\rho)$ in Example 1.



Proof. Part (i). Conditions (a) and (b) ensure that $(F_m(\rho), I_m) \in \mathcal{Q}$. Condition (a) holds by the definition of $F_m(\rho)$. We verify Condition (b) in two cases. If $m \geqslant 1$, then the strict monotonicity of F_m, part (i) of Lemma 3 and $\rho_{m+1} \leqslant \rho < \rho_m$ imply that $R_{m+1}\alpha_{m+1} = F_m(\rho_{m+1}) \leqslant F_m(\rho) < F_m(\rho_m) = R_m\alpha_m$; because $R_{m+1}\alpha_{m+1} \leqslant F_m(\rho) < R_m\alpha_m$, we have $\{i \in N: R_i\alpha_i > F_m(\rho)\} = \{1, \ldots, m\} = I_m$. Alternatively, if $m = 0$, then $R_1\alpha_1 = \rho_1 \leqslant \rho$ and $\{i \in N: R_i\alpha_i > F_m(\rho) = \rho\} = \varnothing = I_0$, so condition (b) holds in both cases.

To see that $(F_m(\rho), I_m)$ is the only pair in $\mathcal{Q}$, consider $(F, I) \in \mathcal{Q}$. By Condition (b) and (18), $I = I_q$ and $F = F_q(\rho)$ for some $q = 0, 1, \ldots, n$, so it remains to show that $q = m$. Again, consider two cases. If $q \geqslant 1$, then condition (b) and part (i) of Lemma 3 imply that $F_q(\rho_{q+1}) = R_{q+1}\alpha_{q+1} \leqslant F_m(\rho) < R_q\alpha_q = F_q(\rho_q)$; hence, the strict monotonicity of F_q assures that $\rho_{q+1} \leqslant \rho < \rho_q$. As $\rho_{m+1} \leqslant \rho < \rho_m$, it follows that necessarily $q = m$. Alternatively, when $q = 0$, condition (b) and $I_0 = \varnothing$ imply that $\rho_1 = R_1\alpha_1 \leqslant F_0(\rho) = \rho$; as $\rho_{m+1} \leqslant \rho < \rho_m$, $m = 0$ follows.

Part (ii). The first conclusion of part (ii) is immediate from part (i), Theorem 1, and the fact that if (F, I) satisfies condition (b), then $I = I_q$ for some (unique) $q \in \{0, 1, \ldots, n\}$. Also, (2), (11), and (12) imply that for $i \in I$, $U_i(x^*) = x_i^*((R_i\alpha_i)/F - 1) = [((R_i\alpha_i - F)F)/(R_i\alpha_i^2)]((R_i\alpha_i)/F - 1) = R_i(1 - F/(R_i\alpha_i))^2$ and for $i \in N\backslash I$, $U_i(x^*) = 0$. Finally, by (6), each $U_i(x)$ is strictly concave in x_i that takes values in a convex set, so no firm will use a randomized strategy that is not pure in a Nash equilibrium. □

Theorem 2 establishes the existence and uniqueness of the Nash equilibrium for this model. These results also follow from the sufficiency conditions of Rosen (1965) without a method for computing it (see §6 for the use of Rosen 1965 to prove existence of a Nash equilibrium in a generalization of the model determined by (2)).

To understand how the unique Nash equilibrium can be calculated explicitly, we reexamine Example 1 and Figure 1. Suppose the interest rate ρ is 0.5. Then $\{i \in N: \rho_i > \rho\} = \{1, 2\}$, implying that the condition in Theorem 2 holds with $m = 2$. The value F that determines the Nash equilibrium (by (11)) is then available from Figure 1 and equals $F = F(\rho) = F_2(\rho) = 3.7748$.

Theorems 1 and 2 imply that the (unique) Nash equilibrium expenditures and utilities can be easily computed from corresponding values of m and F (using (11)). A geometric approach for determining m and F by plotting F versus ρ was just illustrated for Example 1 in Figure 1. The following simple algebraic method returns m for the general problem.

Algorithm 1 (Determining m of Theorem 2)
For $j = 1, \ldots, n+1$:
1. Calculate ρ_j using (21) and (22).
2. Test if $\rho_j \leqslant \rho$. If true or $j = n+1$, let $m = j - 1$ and stop; else iterate for $j + 1$.

The output m satisfies the condition in Theorem 2, so the unique Nash equilibrium expenditures and utilities can be computed using (11) and (27). This algorithm terminates after at most $n + 1$ iterations. Each iteration requires a constant number of arithmetic operations, so a total of $O(n)$ arithmetic operations are required. Also, once m is determined, the computation of $F_m(\rho)$ requires taking only one square root.[2] Finally, the algorithm assumes that the firms are labeled in decreasing order of $R_j\alpha_j$s. If the initial input does not satisfy this assumption, then $R_j\alpha_j$s (or equivalently ρ_js) need to be sorted, which requires $O(n \log n)$ comparisons. However, sorting of all $R_j\alpha_j$s can be avoided by successively using a (linear-time) median-finding algorithm, this reduces the number of comparisons to $O(n)$.

Observations and Insights

Equilibrium Expenditures Bounded by 25% of Revenues. By (11), the Nash equilibrium expenditure x_j^* of firm j is the positive part of a quadratic function of $F \in [0, R_j\alpha_j]$, specifically, $x_j^* = [(F(R_j\alpha_j - F))/(R_j\alpha_j^2)]_+$, where $z_+ = \max\{0, z\}$ for $z \in \mathbb{R}$. This expression attains its maximum value $R_j/4$ at $F = R_j\alpha_j/2$. Consequently, the equilibrium expenditure of a firm never exceeds one quarter of its potential revenue from the project. This surprising bound is independent of all other parameters of the model, i.e., the number of competitors (possibly 0), their attributes, and the interest rate.

Bound on Total Equilibrium Expenditure. The previous paragraph shows that the total equilibrium expenditure (of all firms) $\sum_{i \in N} x_i^*$ is bounded by $\sum_{i \in N} R_i/4$. This subsection provides an alternative bound. If the unique Nash equilibrium is zero, then the total equilibrium expenditure is zero. For a nonzero Nash equilibrium x^*, if $\alpha_u = \min\{\alpha_i\colon i \in N, x_i^* > 0\}$, then the total equilibrium expenditure satisfies

$$\sum_{i\in N} x_i^* \leqslant \frac{\sum_{i\in N}\alpha_i x_i^*}{\alpha_u} = \frac{F-\rho}{\alpha_u} < \left(\frac{F}{\alpha_u R_u}\right) R_u = R_u.$$

Thus, the total equilibrium expenditure cannot exceed the revenue parameter of firm u with the smallest α_i among all firms that invest positively. When α_i is independent of $i \in N$, then R_u can be replaced by $R_v = \min\{R_i\colon i \in N, x_i^* > 0\}$. The forthcoming Example 2 demonstrates that the total equilibrium expenditure can be arbitrarily close to $\max_{i\in N} R_i$.

Active Firms. Theorem 2 shows that the set of firms that will be *active* in the unique Nash equilibrium, i.e., those that invest positive amounts, is determined by the $R_i\alpha_i$s. Specifically, these characteristics determine a threshold value F so that firm j will be active if and only if $R_j\alpha_j > F$. An alternative criterion for determining the set of active firms is through the interest rate ρ, namely, a firm j will be active if and only if $\rho_j > \rho$. In particular, there will be no active firms if $\rho \geqslant \rho_1 = R_1\alpha_1$.

Because the ρ_js are independent of the interest rate ρ, the set of active firms will decrease as the interest rate rises when all other parameters are fixed. To explore the effect of changes in the R_is and α_is on the set of active firms (for fixed ρ), we study the effect of such changes on the ρ_js. Evidently, the ρ_js depend on R_is and α_is only through the product $R_i\alpha_i$. By (20), (19), and Lemma 3, $F_{j-1}(\cdot)$ is independent of $R_i\alpha_i$ for $i \geqslant j$, is increasing and satisfies $F_{j-1}(\rho_j) = R_j\alpha_j$; consequently, ρ_j is an increasing function of $R_j\alpha_j$ and is independent of $R_i\alpha_i$ for $i > j$. Also, by (21) and (19), ρ_j is decreasing in $R_i\alpha_i$ for $i < j$. Hence, a marginal increase in $R_i\alpha_i$ can (1) drive firm j with $j > i$ out of the race, (2) drive firm i into the race. However, it has no effect on whether or not firm j with $j < i$ is active or not.

Definitely Inactive Firms. Call a firm *definitely inactive* if its equilibrium expenditure is zero for all positive interest rates, regardless how small. Our analysis shows that firm j is definitely inactive if and only if $\rho_j \leqslant 0$. As $\rho_1 > \rho_2 = (R_2\alpha_2)^2/(R_1\alpha_1) > 0$, the two firms with highest $R_i\alpha_i$ value can never be definitely inactive; in particular, both firms invest positive amounts when the interest rate exceeds ρ_2.

By (21), firm j will be definitely inactive if and only if $\gamma_j(R_j\alpha_j) \leqslant (j-1)$, or equivalently, $\sum_{i=1}^{j-1}(R_j\alpha_j)/(R_i\alpha_i) \leqslant j-2$. Define the *relative discrepancy of firm j with respect to firm $i<j$* by $\delta_{ij} \equiv (R_i\alpha_i - R_j\alpha_j)/(R_i\alpha_i)$. The condition for a firm to be definitely inactive can then be expressed as $\sum_{i=1}^{j-1}\delta_{ij} = \sum_{i=1}^{j-1}(R_i\alpha_i - R_j\alpha_j)/(R_i\alpha_i) \geqslant 1$, i.e., the cumulative relative discrepancy of firm j is at least one. Clearly, if firm $j \in N$ is not definitely inactive, then firms $k \leqslant j$ are not definitely inactive. Therefore, no firm is definitely inactive if and only if firm n is not definitely inactive, or equivalently, $\sum_{i=1}^{n-1}\delta_{in} = \sum_{i=1}^{n-1}((R_i\alpha_i - R_n\alpha_n)/R_i\alpha_i) < 1$. This is the case if the relative discrepancy between any two firms is sufficiently small, as illustrated by the following example.

Example 2. Let $n \geqslant 2$, $0 < \epsilon \leqslant 1/n^2$, and $R_i\alpha_i = (1-\epsilon)^i$ for each $i \in N$. Recall that $(1-\epsilon)^k \geqslant 1 - k\epsilon$ for all $0<\epsilon<1$ and $k = 1, 2, \ldots$ (which can be shown easily by induction on k). Then,

$$\sum_{i=1}^{n-1}\frac{R_i\alpha_i - R_n\alpha_n}{R_i\alpha_i} = \sum_{i=1}^{n-1}[1-(1-\epsilon)^{n-i}] \leqslant \sum_{i=1}^{n-1}(n-i)\epsilon < n^2\epsilon \leqslant 1,$$

so no firm is definitely inactive, i.e., $\rho_n > 0$ (by (21), $\rho_n = (1-\epsilon)^n[(1-(1-\epsilon)^n)/\epsilon - (n-1)]$). In particular, if $0 < \rho < \rho_n$, then all firms are active in the unique Nash equilibrium. In that case, (20) and $0 < \epsilon \leqslant 1/n^2 < 1$ assure that

$$F > \frac{n-1}{\sum_{i\in N}(1/(1-\epsilon)^i)} \geqslant \frac{n-1}{n/(1-\epsilon)^n} \geqslant \left(\frac{n-1}{n}\right)\left(1-\frac{1}{n^2}\right)^n \geqslant \left(\frac{n-1}{n}\right)\left(1-\frac{n}{n^2}\right) = \left(\frac{n-1}{n}\right)^2.$$

Now, if $R_i = 1$ and $\alpha_i = (1-\epsilon)^i$ for each $i \in N$ and $\rho < 1/n^3 < (n-1)^2/n^3$, then

$$\sum_{i\in N} x_i^* > \sum_{i\in N}\alpha_i x_i^* = F - \rho > \left(\frac{n-1}{n}\right)^2 - \rho > \left(\frac{n-1}{n}\right)^2 - \frac{(n-1)^2}{n^3} = \left(\frac{n-1}{n}\right)^3,$$

implying that as n increases, the lower bound on the total equilibrium expenditure converges to one. Hence, as n increases, the total equilibrium expenditure gets arbitrarily close to the common value of the R_is.

Our analysis shows that the regulation of the interest rate by a centralized decision maker (say, a governmental agency) may affect the set of active firms in the race, e.g., by offering loans at reduced interest rates, leading to a larger set of active firms. In particular, such an action always assures that there will be active competition between at least two firms. However, definitely inactive firms cannot be induced to enter the race just by the adjustment of the interest rate.

Expenditures Without Competition vs. Equilibrium Expenditures. If firm j operates in an isolated environment without any competition, then its utility from investing $x_j \geqslant 0$ in the project is $U_j(x_j) = (R_j\alpha_j x_j)/(\alpha_j x_j + \rho) - x_j$. Its optimal investment is then

$$x_j^* = \begin{cases} \dfrac{\sqrt{R_j\alpha_j\rho} - \rho}{\alpha_j} & \text{if } R_j\alpha_j > \rho, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

In particular, the criterion for positive investment is $\rho < R_j\alpha_j$. We next observe that

$$\rho_j = \sum_{i=1}^{j} \frac{(R_j\alpha_j)^2}{R_i\alpha_i} - (j-1)R_j\alpha_j$$
$$\leqslant jR_j\alpha_j - (j-1)R_j\alpha_j = R_j\alpha_j,$$

with equality holding if and only if $j = 1$. So, for the firm with the highest $R_i\alpha_i$ value, the criterion for positive investment is the same whether or not there is competition. However, for the remaining firms, the presence of competition induces a tighter criterion for getting actively involved in the race than when they act on their own.

Appearance of a New Firm. Suppose a new firm $k \notin N$ enters the market. The threshold ρ_m for a firm m with $R_m\alpha_m > R_k\alpha_k$ is not affected by the appearance of the new firm. However, ρ_m for a firm m with $R_m\alpha_m < R_k\alpha_k$ changes as follows:

$$\rho_m^{\text{new}} = \left(\gamma_m + \frac{1}{R_k\alpha_k}\right)(R_m\alpha_m)^2 - mR_m\alpha_m$$
$$= \rho_m^{\text{old}} + (R_m\alpha_m)\left[\frac{R_m\alpha_m}{R_k\alpha_k} - 1\right] < \rho_m^{\text{old}}$$

(with the obvious interpretation for notation). Consequently, the criterion for entering the race tightens and a firm will never enter the race if it was not in it before the arrival of the new firm. However, it is possible that a firm that was active before will now quit the race (this occurs if and only if $\rho_m^{\text{new}} \leqslant \rho < \rho_m^{\text{old}}$). Therefore, the appearance of a new firm into the market works against the "weaker" firms. This observation may lead to the conclusion that the equilibrium expenditure of any firm will decrease when a new firm enters the market. However, the following example shows that the appearance of either a "weak" or "strong" firm can increase or decrease another firm's equilibrium investment. Example 3 and the forthcoming Example 4 also illustrate the unpredictable effect of changes in other parameters on equilibrium expenditures.

EXAMPLE 3. Suppose initially $n = R_1 = \alpha_1 = 1$ and $\rho = 0.01$. The equilibrium expenditure of firm 1 is $x_1^* = 0.09$, by (24). If a second firm with $R_2 = 0.9$ and $\alpha_2 = 1$ joins the market, then by (21), $\rho_1 = 1$ and $\rho_2 = 0.81$, so both firms will be active in the race when $\rho = 0.01$. The Equation (10) gives $F = 0.4835$ and $x_1^* = 0.2497 > 0.09$. If the parameters of the new firm are $R_2 = 1.1$ and $\alpha_2 = 1$, then (noting that (18) is violated) $\rho_2 = 1.1$ and $\rho_1 = 0.9091$, so both firms will be active with $F = 0.5336$ and $x_1^* = 0.2489 > 0.09$.

If the interest rate is instead $\rho = 0.5$, then $x_1^* = 0.2071$ when firm 1 operates alone. The arrival of a second firm with $R_2 = 0.9$ and $\alpha_2 = 1$ yields $\rho_1 = 1$, $\rho_2 = 0.81$; both firms will be active, $F = 0.7781$ and $x_1^* = 0.1727 < 0.2071$. On the other hand, if $R_2 = 1.1$ and $\alpha_2 = 1$, then $\rho_2 = 1.1$, $\rho_1 = 0.9091$; both firms will be active, $F = 0.8368$ and $x_1^* = 0.1366 < 0.2071$.

Ranking of Active Firms by the Expected Completion Time. When firm j invests $x_j > 0$, its expected completion time for developing the product is $1/(\alpha_j x_j)$. Let x^* be the (unique) Nash equilibrium. Corollary 1 implies that among firms that enter the race, the ranking by $R_j\alpha_j$, from high to low, is the same as the ranking by the equilibrium expected development times $1/(\alpha_j x_j^*)$, from short to long. In particular, when the interest rate goes up, firms that leave the race have longer expected development times than the ones that stay.

The sensitivity of the equilibrium investments and utilities to changes in various system parameters is discussed in Appendix A.

4. Global Optimality

This section considers the scenario where a single (centralized) decision maker controls which of the n firms will invest and how much each active firm will invest. Such "parallel funding" has been explored in the OR literature, e.g., Gerchak and Kilgour (1999) and Gürler et al. (2000) analyze situations in which there is a predetermined threshold that determines the success of the R&D project. By authorizing the simultaneous development of a product by multiple independent teams, these models maximize the probability that the best team will exceed the threshold. However, there is no such predetermined threshold in our setting.

The aggregate utility gained by all firms under a vector x that represents the firms' expenditure levels, referred to as the *global utility of* x, is given by

$$U(x) = \sum_{i=1}^{n} U_i(x) = \frac{\sum_{i=1}^{n} R_i\alpha_i x_i}{\alpha^T x + \rho} - x^T 1 \quad \text{for } x \in \mathbb{R}_{\oplus}^n. \tag{25}$$

A vector $x^\star \in \mathbb{R}_{\oplus}^n$ is called *globally optimal* if it maximizes $U(x)$ over $\mathbb{R}_{\oplus}^n$, i.e.,

$$x^\star \in \arg\max_{x \in \mathbb{R}_{\oplus}^n} U(x). \tag{26}$$

Call a vector $x \in \mathbb{R}_{\oplus}^n$ *i-monopolistic* if $x_i > 0$ and $x_j = 0$ for all $j \in N \backslash \{i\}$; call $x \in \mathbb{R}_{\oplus}^n$ *monopolistic* if it is i-monopolistic for some $i \in N$. The next result derives an explicit globally optimal solution that is monopolistic when $\max_{i\in N} R_i\alpha_i > \rho$ and is zero otherwise.

THEOREM 3. (i) *If* $\max_{i\in N} R_i\alpha_i \leqslant \rho$, *then* $x^\star = 0$ *is a unique globally optimal solution.*

(ii) *If* $\max_{i\in N} R_i\alpha_i > \rho$ *and* $i^\star$ *maximizes* $\sqrt{R_i} - \sqrt{\rho/\alpha_i}$ *over* $i \in N$, *then a globally optimal solution is given by*

$$x_i^\star = \begin{cases} \dfrac{\sqrt{\rho R_{i^\star}\alpha_{i^\star}} - \rho}{\alpha_{i^\star}} & \text{if } i = i^\star, \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

with global utility $U(x^\star) = (\sqrt{R_{i^\star}} - \sqrt{\rho/\alpha_{i^\star}})^2$; *further, if* $i^\star$ *is a unique maximizer, then* (27) *defines a unique globally optimal solution.*

Proof. Part (i). For $x \in \mathbb{R}^n_\oplus$, let $v(x) \equiv \alpha^T x$; then $U(x) = \sum_{i\in N} x_i[(R_i\alpha_i)/(v(x)+\rho) - 1]$. For $x \in \mathbb{R}^n_\oplus \backslash \{0\}$, $v(x) > 0$. Therefore, if $\max_{i\in N} R_i\alpha_i \leqslant \rho$, then $U(x) = \sum_{i\in N} x_i[(R_i\alpha_i)/(v(x)+\rho)-1] < 0 = U(0)$ for $x \in \mathbb{R}^n_\oplus\backslash\{0\}$, and (i) follows.

Part (ii). Suppose $\max_{i\in N} R_i\alpha_i > \rho$ and let $X_v \equiv \{x \in \mathbb{R}^n_\oplus\colon v(x) = v\}$ for $v \geqslant 0$. Then,

$$\max_{x\in\mathbb{R}^n_\oplus} U(x) = \max_{v\geqslant 0}\Big[\max_{x\in X_v} U(x)\Big] = \max_{v\geqslant 0}\left\{\max_{x\in X_v}\sum_{i\in N} x_i\left[\frac{R_i\alpha_i}{v+\rho} - 1\right]\right\}. \tag{28}$$

For each $v > 0$ and $i \in N$, there is an i-monopolistic optimal vector $x^i_v \in X_v$. In particular, by standard results in linear programming, the linear function $\sum_{i\in N} x_i[(R_i\alpha_i)/(v+\rho)-1]$ attains a maximum over X_v at a vertex. As the vertices of X_v are the x^i_vs, we get that

$$\max_{x\in X_v}\sum_{i\in N} x_i\left[\frac{R_i\alpha_i}{v+\rho}-1\right] = \max_{i\in N}\frac{v}{\alpha_i}\left(\frac{R_i\alpha_i}{v+\rho}-1\right). \tag{29}$$

As (29) extends to $v = 0$, it follows from (28) that

$$\max_{x\in\mathbb{R}^n_\oplus} U(x) = \max_{v\geqslant 0}\left[\max_{i\in N}\frac{v}{\alpha_i}\left(\frac{R_i\alpha_i}{v+\rho}-1\right)\right] = \max_{i\in N}\left[\max_{v\geqslant 0}\frac{v}{\alpha_i}\left(\frac{R_i\alpha_i}{v+\rho}-1\right)\right]. \tag{30}$$

The internal maximum on the right-hand side of (30) is attained at $v = (\sqrt{R_i\alpha_i\rho}-\rho)_+$ with the objective function value $[(\sqrt{R_i\alpha_i\rho}-\rho)/\alpha_i]_+[(R_i\alpha_i)/\sqrt{R_i\alpha_i\rho}-1] = [(\sqrt{R_i}-\sqrt{\rho/\alpha_i})_+]^2$. As $\max_{i\in N} R_i\alpha_i > \rho$, the external maximum on the left-hand side of (30) is attained at $i^\star \in \arg\max_i(\sqrt{R_i}-\sqrt{\rho/\alpha_i})$ with the objective function value $U(x^\star) = (\sqrt{R_{i^\star}}-\sqrt{\rho/\alpha_{i^\star}})^2 > 0$, where $x^\star$ is the $i^\star$-monopolistic vector given by (27). Thus, $\max_{x\in\mathbb{R}^n_\oplus} U(x) = U(x^\star) > 0$.

Finally, if $i^\star$ is a unique maximizer of $\sqrt{R_i}-\sqrt{\rho/\alpha_i}$ over $i \in N$, then for $z \in \mathbb{R}^n_\oplus\backslash\{0\}$, $U(z) \leqslant \max_{i\in N} U(x^i_{v(z)}) \leqslant U(x^\star)$, and it is easily verified that one of the two inequalities is strict whenever $z \neq x^\star$. □

Because $U(\cdot)$ is not necessarily concave, a solution satisfying the KKT conditions need not be globally optimal, but the proof of Theorem 3 does not rely on the KKT conditions.

Theorem 3 implies that a globally optimal solution will lead to either lack of activity or to a monopoly. When a monopolistic solution exists, the active firm is determined by maximizing $\sqrt{R_i}-\sqrt{\rho/\alpha_i}$ over $i \in N$, which implies that this firm has relatively high values of R_i and of α_i as compared to the other firms. We note that the criterion $R_i\alpha_i$ used for ranking firms in §3 for constructing the Nash equilibrium also gives priority to firms with high values of R_i and of α_i (but Example 4 below demonstrates that the balance is different). In particular, if there is a firm i with $R_i\alpha_i > \rho$ that maximizes both R_j and α_j individually over $j \in N$, then this firm can be selected as the monopolistic firm $i^\star$ in Theorem 3. Also, because the expression that $i^\star$ is to maximize equals $(1/\sqrt{\alpha_i})(\sqrt{R_i\alpha_i}-\sqrt{\rho})$, if a firm i with $R_i\alpha_i > \rho$ maximizes $R_j\alpha_j$ and at the same time minimizes α_j over $j \in N$, then firm i can be selected as $i^\star$. In either of these two special cases, the monopolistic firm maximizes $R_i\alpha_i$ and is active in the Nash equilibrium (derived in §3).

In contrast to the globally optimal solution, our analysis in §3 shows that the number of active firms in a Nash equilibrium can range from 0 to n; in fact, if the $R_i\alpha_i$s are close to each other, no firm will be definitely inactive and for each $0 \leqslant k \leqslant n$, there exists an interest rate for which k firms will be active in the equilibrium. The following example demonstrates that it is possible that a unique monopolistic firm of a globally optimal solution need not be active in the unique Nash equilibrium with active firms.

Example 4. Consider two firms such that $R_1 = \alpha_1 = 1$, $R_2 = 10$ and $\alpha_2 = 0.09$, and let $\rho = 0.85$. Using the notation introduced in §3, $\rho_1 = R_1\alpha_1 = 1 > 0.85 = \rho$, $R_2\alpha_2 = 0.9 > 0.85 = \rho$, $\rho_2 = (R_2\alpha_2)^2/R_1\alpha_1 = 0.81 < 0.85 = \rho$. Therefore, by Theorem 2, only firm 1 will be active in the Nash equilibrium. Because $R_i\alpha_i > \rho$ for $i = 1, 2$, both firms are candidates to be the monopolistic firm in a globally optimal solution. As $\sqrt{R_1}-\sqrt{\rho/\alpha_1} = 1-\sqrt{0.85} = 0.0780 < 0.0890 = \sqrt{R_2}-\sqrt{\rho/\alpha_2}$, Theorem 3 implies that the monopolistic solution with $i^\star = 2$ is a unique globally optimal solution.

Example 4 suggests that the Nash equilibrium can be substantially different from the globally optimal solution. However, Theorems 2 and 3 together imply that the global utility of the (unique) Nash equilibrium is positive if and only if the optimal global utility is positive and this happens if and only if $\max_{i\in N} R_i\alpha_i > \rho$.

The next corollary shows that the solution determined in Theorem 3 (which maximizes the global utility given by (25)) is also optimal when the global utility is defined as the maximum utility over all players.

Corollary 2. *The solution $x^\star$ in Theorem 3 maximizes $\tilde{U}(x) \equiv \max_{i\in N} U_i(x)$ over $x \in \mathbb{R}^n_\oplus$.*

Proof. Consider $x \in \mathbb{R}^n_\oplus$. If $\max_{i\in N} R_i\alpha_i \leqslant \rho$, then

$$\tilde{U}(x) = \max_{i\in N} x_i\left(\frac{R_i\alpha_i}{\alpha^T x+\rho}-1\right) \leqslant \max_{i\in N} x_i\left(\frac{\rho}{\alpha^T x+\rho}-1\right) \leqslant 0 = \tilde{U}(x^\star).$$

Alternatively, if $\max_{i\in N} R_i\alpha_i > \rho$, then by Theorem 3, for any $x \in \mathbb{R}^n_\oplus$,

$$\tilde{U}(x) = \max_{i\in N} U_i(x) \leqslant \sum_{i\in N} U_i(x) = U(x) \leqslant U(x^\star) = U_{i^\star}(x^\star) \leqslant \max_{i\in N} U_i(x^\star) = \tilde{U}(x^\star). \quad \square$$

We say that *technology substitution is feasible* if the centralized decision maker can select and match the characteristics $R_{\bullet}$ and $\alpha_{\bullet}$ from different firms. In effect, this means that there are n^2 firms competing, corresponding to the pairs (R_u, α_v) with $(u, v) \in N \times N$. Of course, any pair $(u^\star, v^\star) \in [\arg\max_{i\in N} R_i] \times [\arg\max_{i\in N} \alpha_i]$ maximizes both $R_{\bullet}$ and $\alpha_{\bullet}$; consequently, when $(\max_{i\in N} R_i)(\max_{i\in N} \alpha_i) > \rho$, the selection of any such a pair as a monopoly with investment of $x^\star_{u^\star v^\star} = (\sqrt{\rho R_{u^\star}\alpha_{v^\star}} - \rho)/\alpha_{v^\star}$ is optimal under both U and $\tilde{U}$. In the alternative case, $x^\star = 0$ is the only optimal solution.

The gap between the equilibrium and optimal global utilities is discussed in Appendix B.

5. Numerical Experiment

The results so far rely on the assumption that all the firms assess their expected revenues correctly; however, in practice, there may be times when these values do not match the reality once the project is completed. To understand this phenomenon and study the robustness of the Nash equilibrium solution to such errors, we ran the following experiment in MATLAB. Let n denote the number of firms in the market and set the technology parameter of each firm to one. Each firm $i \in N$ is assigned a revenue interval $[a_i, b_i]$ with $b_i = a_i + c$, $a_i = \lceil d b_{i-1} \rceil$ and $a_n = 1$, where $\lceil \cdot \rceil$ denotes the ceiling function, and c and d are positive constants. Letting $d > 1$ assures that the intervals are disjoint, and so a change in the revenue of firm i over its corresponding interval does not affect the ranking of the firms with respect to $R_i\alpha_i$. In each replication of the experiment, the revenue R_j of firm j is chosen uniformly at random from its corresponding interval and it is assumed that the firms make their decisions based on these values. Then, for some $i \in N$, another revenue value, say $\tilde{R}_i$, is determined uniformly at random from its interval. We computed the equilibrium utility of firm i with all firms' investments based on the true revenue $\tilde{R}_i$ (denoted as $x(\tilde{R}_i)$) and the realized utility of firm i with all firms' investments based on the assessment R_i (denoted as $x(R_i)$). We define *regret* $\Lambda_i(\cdot)$ and *relative regret* $\lambda_i(\cdot)$ of firm i as follows:

$$\Lambda_i(R_i, \tilde{R}_i) = \left[\frac{\tilde{R}_i\alpha_i x_i(\tilde{R}_i)}{\alpha^T x(\tilde{R}_i) + \rho} - x_i(\tilde{R}_i)\right] - \left[\frac{\tilde{R}_i\alpha_i x_i(R_i)}{\alpha^T x(R_i) + \rho} - x_i(R_i)\right] \quad (31)$$

$$\lambda_i(R_i, \tilde{R}_i) = \frac{\Lambda}{(\tilde{R}_i\alpha_i x_i(\tilde{R}_i))/(\alpha^T x(\tilde{R}_i) + \rho) - x_i(\tilde{R}_i)}. \quad (32)$$

These quantities can be negative, as the following example illustrates.

Example 5. Consider a two-firm market with $\alpha_1 = \alpha_2 = 1$, $\rho = 0.1$ and $R_1 = 13$. Suppose firm 2 estimates its revenue to be $R_2 = 2$. The equilibrium expenditures of the firms with this assessment is $x_1 = 1.5711$ and $x_2 = 0.1571$. If the true revenue of firm 2 turns out to be $\tilde{R}_2 = 3$, then its realized utility becomes 0.1007. If the firms were to invest according to the value $\tilde{R}_2$, then the first firm would invest 2.0399, whereas the second one would invest 0.3938 and earn a utility of 0.0725, which is less than what it earns with the incorrect assessment of its revenue. Hence, in this case, firm 2 has a negative regret, i.e., it can benefit from wrongly estimating its own revenue. Similar examples show that firms may sometimes benefit from inaccurate assessments of revenues by other firms.

Table 1 provides summary statistics about the relative regret resulting from a 1,000-replication run of this experiment, with $\rho = 0.1$, $c = 49$, $d = 1.1$, and $i = 1$. We focused on the first firm to avoid division by zero, because the first firm is always active for sufficiently small interest rates.

Table 1 suggests that the relative regret is negligible in this setup as far as the strongest firm is concerned. Furthermore, the relative regret seems to be diminishing as the number of firms increases.

Table 1. Statistics of relative regret.

Number of firms	2	5	10	15	20	50
Maximum	0.1247	0.0849	0.0498	0.0282	0.0187	0.0010
Sample mean	0.0049	0.0012	−0.0008	−0.0003	−0.0001	−0.0000
Sample variance	0.0014	0.0010	0.0004	0.0001	0.0001	0.0000

6. Discussion, Extensions, and Future Work

This paper addresses a competitive situation that is becoming more and more common in technology markets where multiple firms compete over the development of cutting-edge products and services. To conclude, we discuss several variants of our model and the extension of our results to some of these.

Relations to Existing Models

Our analysis depends solely on the expression of $U_i(x)$s as in (2) and not on their derivations from the exponentiality assumption and the winner-takes-all structure. In fact, scaling of the input variables enables the conversion of any expression of the form $U_i(x) = (a_i x_i)/(b^T x + \rho) - c_i x_i$ with $a, b, c \in \mathbb{R}^n_+$, $a \equiv (a_i)$ and $c \equiv (c_i)$ to (2).

Models to which our results apply include the basic rent-seeking model in the public choice literature and an instance of the well-known Cournot model in economics.

• **Rent seeking**. Clark and Riis (1998) define a *contest* as a competition among players over a single prize, where each player makes an investment that cannot be withdrawn once committed, and this investment enhances the winning probability of its investor while reducing the chances of the other players. The function mapping the investments to the winning probabilities is called *contest success functions*. As an extension of the work by Skaperdas (1996) to contests with nonidentical players, Clark and Riis (1998) prove that the probability $p_i(x)$ that player $i \in N$ wins the contest when the investment of the participants is represented by the vector x is given by $p_i(x) = (\alpha_i x_i^r)/(\sum_{j\in N} \alpha_j x_j^r)$ for some positive constants $r, \alpha_1, \ldots, \alpha_n$ if and only if the following four axioms are satisfied:

1. Probability: $0 \leqslant p_i(x) < 1$ and $\sum_{i\in N} p_i(x) = 1$; if $x_i > 0$, then $p_i(x) > 0$.
2. Monotonicity: $p_i(x)$ is strictly increasing in x_i and nonincreasing in x_j for $j \neq i$.
3. Independence: $p_i(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) = p_i(x)/(1-p_j(x))$ for $j \neq i$.
4. Homogeneity: $p_i(x) = p_i(ax)$ for $a > 0$.

These functions are commonly used in modeling tournaments, conflict and rent seeking; see Skaperdas (1996), Szidarovszky and Okuguchi (1997), and references therein. For $r = 1$, the utility of player i can be expressed by (2) with $\rho = 0$.

• **Cournot equilibrium.** A Cournot equilibrium is a Nash equilibrium where each firm $i \in N$ maximizes its utility $\tilde{U}_i(y) = y_i[P(y^T 1) - c_i]$ with respect to its production quantity $y_i \geqslant 0$. The function P represents the unit price of the good in terms of the total supply, and $c_i > 0$ is the unit production cost of firm i. We observe that the Cournot model with $P(z) = (z+\rho)^{-1}$ for some $\rho \geqslant 0$ is equivalent to our model. Specifically, by letting $\alpha_i = 1$ and $R_i = c_i^{-1}$ in (2), the utility of firm i in the winner-takes-all model satisfies $U_i(x) = c_i^{-1}\tilde{U}_i(x)$ and the Nash equilibria for the corresponding winner-takes-all model are precisely the Cournot equilibria. Conversely, the Cournot equilibria with $P(z) = (z+\rho)^{-1}$, $c_i = (R_i\alpha_i)^{-1}$ and $y_i = \alpha_i x_i$ are precisely the Nash equilibria for the winner-takes-all model. However, a globally optimal solution of the winner-takes-all model maximizes $\sum_{i\in N} U_i(x) = \sum_{i\in N}[(\alpha_i c_i)^{-1}\tilde{U}_i(y)]$ rather than $\sum_{i\in N}\tilde{U}_i(y)$, which can be regarded as the aggregate utility of the Cournot model. In particular, the globally optimal solutions to these two problems need not be the same unless $R_i = R$ for all $i \in N$.

Existence and uniqueness of Cournot equilibrium for general price functions have been widely studied in the economics literature, e.g., see Kolstad and Mathiesen (1987) for necessary and sufficient conditions for uniqueness; however, the generality of the price function does not usually allow the explicit derivation of the equilibrium quantities and payoffs. A recent paper by Harris et al. (2010) provides explicit solutions for Cournot models in which the price function is twice differentiable, is decreasing, and has a saturation point, i.e., $P(z) = 0$ for some $z \geqslant 0$, but the price function $P(z) = (z+\rho)^{-1}$ that converts the winner-takes-all model we study to a Cournot model does not have such a saturation point.

Zero Interest Rate

Our results can be extended to the case $\rho = 0$. This assumption appeared in variants of our model considered in earlier studies that did not derive explicit solutions, e.g., Harris and Vickers (1985), Gerchak and Parlar (1999). Also, the basic rent-seeking model introduced by Tullock (1980) does not consider discounting. If $\rho = 0$, then the utility of firm $i \in N$ is

$$U_i(x) = \begin{cases} \dfrac{R_i\alpha_i x_i}{\alpha^T x} - x_i & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{33}$$

These functions are not continuous at $x = 0$,[3] which is not a Nash equilibrium, because when all firms invest 0, any firm i benefits from slightly increasing its investment to $0 < t < R_i$. Its utility then increases to $R_i - t > 0 = U_i(0)$. Therefore, if there is a Nash equilibrium, it has to be in the set $\mathbb{R}^n_\oplus \setminus \{0\}$. Because each $U_i(x)$ is concave in x_i on this domain, Lemma 2 remains valid when (8) is augmented with $x^* \neq 0$. Also, Theorem 2 remains valid, except that I is required to be nonempty and when $|I| = 1$, the positivity of F is dropped (in this case, 0 is a unique solution of (10)). Also, Lemma 3 (which does not depend on ρ) and the observation that $\rho_2 > 0$ (when $n \geqslant 2$) remain valid.

When $n \geqslant 2$, the assertion and conclusions of Theorem 2 hold with $2 \leqslant m \leqslant n$ (instead of $0 \leqslant m \leqslant n$), establishing the existence of a unique Nash equilibrium that can be computed explicitly. Furthermore, the integer m in the modification of Theorem 2 is the first integer satisfying $\gamma_m - (m-1)R_m\alpha_m \leqslant 0$ (and Lemma 3 assures that $\gamma_m - (m-1)R_m\alpha_m$ is decreasing in m). Also, (20) simplifies to $F_m(0) = \gamma_m/(m-1)$. For $n \geqslant 1$, $\sup_{x\in\mathbb{R}^n_\oplus} U(x) = \max_{i\in N} R_i$ cannot be attained, so there is no globally optimal solution (and no Nash equilibrium for $n = 1$).

Spreading Expenditures/Payments

The utility functions formulated in §2 assume that capital investment is fixed and paid up front. If, in fact, the fixed investment cost is spread over a finite horizon, say T_i (with commitment to continue to pay even if the competition is lost), then the discounted investment cost would be $\int_0^{T_i}(x_i/T_i)e^{-\rho t}\,dt = \beta_i x_i$ for a corresponding positive constant β_i; the second term of the utility of function of firm i will then be $-\beta_i x_i$ rather than $-x_i$. By changing the variables to $x_i' \equiv \beta_i x_i$ and letting $\alpha_i' \equiv \alpha_i/\beta_i$, we obtain the utility function in (2); hence, our results will apply with α_i's replacing the α_is.

Renewable Resources

Resources that are invested in developing a technology may be "renewable," in which case a commitment is made for continuous expenditure at a fixed rate throughout the lifetime of the project (whether ending due to winning or losing the competition). If the predetermined investment rate is x_i, yielding a parameter $\alpha_i x_i$ of the exponential distribution of the completion time, then the utility function of firm i would be $(R_i\alpha_i x_i - x_i)/(\alpha^T x + \rho)$ and a Nash equilibrium need not exist—firm i will try to invest as much as possible if $R_i\alpha_i > 1$ and none otherwise. More interesting situations arise when the investment rate can vary across time. Milgrom and Roberts (1990), Veinott (1989), and Vives (2005) considered the case of renewable resources under more general cost functions and derived results about existence and sensitivity without providing explicit solutions to the problem or an efficient way to compute them.

Nonlinear Rates of Exponential Distributions

Throughout this paper, we assumed that the completion time of a firm is exponentially distributed with a rate that is linear in the investment of the firm, but the existence of a Nash equilibrium can be extended to nonlinear rates. Suppose the rate of the completion time for firm i is given by a nonlinear function g_i of its investment x_i, then (2) becomes

$$U_i(x) = \frac{R_i g_i(x_i)}{\sum_{j=1}^n g_j(x_j) + \rho} - x_i. \tag{34}$$

Because each g_i is nonnegative, firm i will not invest more than R_i in a Nash equilibrium, so x_i takes values in an effectively bounded convex set. Consequently, Rosen's (1965) result regarding the existence of a pure-strategy Nash equilibrium can be applied whenever $U_i(x)$ is strictly concave in x_i for all $i \in N$. This is true when g_i is strictly concave in $x_i \in [0, R_i]$, e.g., $g_i(x_i) = \alpha_i x_i^k$ with $\alpha_i > 0$ and $0 < k < 1$. The case where each completion time has a Weibull distribution with shape parameter $0 < k < 1$ can be formulated as such. However, with the nonlinear rates, uniqueness requires further conditions on the g_is and obtaining an explicit representation is rather difficult.

If $k > 1$ in (34), then strict concavity is not guaranteed; nevertheless, the existence of a Nash equilibrium over the domain of randomized strategies follows from Glicksberg (1952) and the finiteness of the support for every equilibrium strategy follows from Karlin (1959, Theorem 7.1.1). Identifying the support of each firm's Nash equilibrium strategy is an ongoing research at the time being.

Limited Resources and Multiple Projects

This paper addressed a competition over a single project. A more challenging problem would allow competition among firms over multiple projects, leading to multiple simultaneous R&D races, see Gerchak and Parlar (1999). Such multiple races cannot be handled separately (by straightforward extensions of our results) when the firms have a finite capacity of resources to allocate to different projects; consequently, the decisions of the firms are subject to constraints on their total expenditures. A forthcoming paper treats this issue by imposing budgetary constraints on the firms.

Imperfect or Incomplete Information

The competitive problem becomes more complex when firms have access to private information. For example, firms may experience the phenomenon of "winner's curse," explaining situations where the benefits that firms gain from winning the race for a patentable technology do not match the expenses they had in developing (e.g., Pharmaceutical Research and Manufacturers of America 2011).

Appendix A. Sensitivity Analysis

A.1. Equilibrium Utilities and Expenditures

The analysis of the set of active firms in the competitive setting shows that an increase in the interest rate or in the parameters of other firms can drive an active firm out of the race but cannot drive it into the race. An increase in its own parameters can make an inactive firm active, but will not make an active firm inactive. Intuition suggests that the equilibrium utilities and expenditures follow corresponding monotonic trends with such changes. The following development shows that some of these speculations hold whereas others do not.

Let x^* be the unique Nash equilibrium with $U_i^* \equiv U_i(x^*)$ the equilibrium utility of firm $i \in N$. To analyze the sensitivity of these quantities to changes in a parameter z, we parameterize the system characteristics by z, i.e., we write $U_i^*(z)$, $x_i^*(z)$, $F(z)$, $F_m(\rho, z)$, and $I(z)$. In particular, Theorems 1 and 2 show that

$$U_i^*(z) = \begin{cases} R_i\left(1 - \dfrac{F(z)}{R_i\alpha_i}\right)^2 & \text{if } i \in I(z), \\ 0 & \text{otherwise,} \end{cases} \tag{A1}$$

$$x_i^*(z) = \begin{cases} \dfrac{F(z)}{\alpha_i} - \dfrac{F^2(z)}{R_i\alpha_i^2} & \text{if } i \in I(z), \\ 0 & \text{otherwise.} \end{cases} \tag{A2}$$

When referring to a parameter symbolically, we use quotations, e.g., "z."

Lemma 4. *Let "z" stand for "ρ," "α_j," or "R_j" for some $j \in N$. Then $F(z)$ is continuous and piecewise differentiable. Furthermore, $F(\rho)$ is increasing and when "z" stands for "α_j" or "R_j," if $F(z)$ is differentiable at z and $j \in I(z)$, then $F'(z) > 0$ and $F(z) - zF'(z) > 0$.*

Proof. The analysis of the effect of the system parameters on the set of active firms in the unique Nash equilibrium in §3 shows that $I(z)$ is piecewise constant.

We first let "z" stand for "ρ." Consider a ρ-interval where $I(\rho)$ is constant, say, $I(\rho) = I(m)$. Then $F(\rho) = F_m(\rho)$ (the equality following from Theorem 2) is given by (20), which assures that $F(\rho)$ is continuous, increasing, and differentiable, the latter on the interior of the given ρ-interval. The endpoints of the given "ρ"-interval are $\rho_m(z)$ and $\rho_{m+1}(z)$, and the global continuity of $F(\cdot)$ follows directly from Lemma 3.

Next, let "z" stand for "α_j" or "R_j" and consider a z-interval where $I(z)$ is constant and contains j, say $I(z) = I_m$. Again, $F(z) = F_m(\rho, z)$ is expressed by (20), hence continuous and differentiable, the latter on the interior of the given z-interval. To see that $F(\cdot)$ is globally continuous, observe that the endpoints of the (closure of the) "z"-interval where $I(z) = I_m$ have $\rho_m(z) = \rho$ and $\rho_{m+1}(z) = \rho$, respectively. Consider the point $\hat{z}$ with $\rho_m(\hat{z}) = \rho$, the other case following from similar arguments. By Lemma 3, $F_m(\rho,\hat{z}) = F_m[\rho_m(\hat{z}),\hat{z}] = R_m\alpha_m = F_{m-1}[\rho_m(\hat{z}),\hat{z}] = F_{m-1}(\rho,\hat{z})$.

Next assume that "z" is in the interior of the z-interval where $I(z) = I_m$ (assuring that $F(\cdot)$ is differentiable at z) and that $j \in I(z)$. Consider a quadratic equation $ay^2 + by + c = 0$. Let $y(a)$ denote the larger root of the equation (as a function of $a < b^2/(4c)$, with b and c fixed). Then $y(\cdot)$ is differentiable and $y^2(a) + 2ay(a)y'(a) + by'(a) = [ay^2(a) + by(a) + c]' = 0$; hence,

$$y'(a) = \frac{-y^2(a)}{2ay(a)+b}.$$

The quadratic equation defining $F(z) = F_m(z)$ has $a = \gamma_m(z)$ (with $\gamma_m(z)$ given by (19)), $b = m-1$, $y(a) = F(z)$, and $2ay(a) + b = \sqrt{(m-1)^2 + 4\gamma_m(z)\rho}$. As $\gamma'_m(z) = -1/(R_j\alpha_j z)$,

$$F'(z) = \frac{-F^2(z)\gamma'_m(z)}{\sqrt{(m-1)^2+4\gamma_m(z)\rho}} = \frac{F^2(z)}{R_j\alpha_j z\sqrt{(m-1)^2+4\gamma_m(z)\rho}} > 0.$$

Next note that if $m = 1$, then $j = 1$, $F(z) = \sqrt{\rho R_1\alpha_1}$, and $F(z) - zF'(z) = (\sqrt{\rho R_1\alpha_1})/2 > 0$. Alternatively, if $m \geqslant 2$, then $\sqrt{(m-1)^2+4\gamma_m(z)\rho} \geqslant m-1 \geqslant 1$ and (as $j \in I(z)$ assures $R_j\alpha_j > F(z)$),

$$F(z) - zF'(z) = F(z)\left[1 - \frac{F(z)}{R_j\alpha_j\sqrt{(m-1)^2+4\gamma_m(z)\rho}}\right] \geqslant F(z)\left[1 - \frac{F(z)}{R_j\alpha_j}\right] > 0. \quad \square$$

We now explore the dependency of firm i's equilibrium expenditure and utility (given by (A1) and (A2)) on various system characteristics. When i is not active, the sensitivity analysis is trivial. Hence, we assume through the end of this subsection that i is active. Also, (A1) and (A2) assure that i's equilibrium expenditure and utility inherit the continuity and piecewise differentiability of $F(\cdot)$ (established in Lemma 4); hence, our analysis of their dependencies on a parameter "z" can and will be restricted to (open) "z"-intervals where F is differentiable.

Aggressive and Regressive Positions. The forthcoming sensitivity analysis reveals that the sign of $R_i\alpha_i - 2F$ determines the direction of change in the equilibrium expenditure of firm i in many cases. We say that an active firm i is in an *aggressive* position if $R_i\alpha_i > 2F$ and in a *regressive* position otherwise. For $n > 1$, the weakest active firm (i.e., the one with the smallest $R_j\alpha_j$ among all active firms) in the unique Nash equilibrium is always in a regressive position, because

$$2F = \frac{(i-1)+\sqrt{(i-1)^2+4\gamma_i\rho}}{\gamma_i} > \frac{i}{\gamma_i} \geqslant R_i\alpha_i.$$

We will see that for some (small) parameter changes, firms in aggressive (regressive) position increase (decrease) their equilibrium investments when they face a tougher environment.

Interest Rate ρ. The assumption that i is active is equivalent to $\rho < \rho_i$. For such ρ, (A1) and (A2) imply that

$$\frac{dU_i^*}{d\rho}(\rho) = \frac{-2F'(\rho)}{\alpha_i}\left[1 - \frac{F(\rho)}{R_i\alpha_i}\right] \quad \text{and}$$
$$\frac{dx_i^*}{d\rho}(\rho) = \frac{F'(\rho)}{\alpha_i}\left[1 - \frac{2F(\rho)}{R_i\alpha_i}\right].$$

As $F'(\rho) > 0$ (by Lemma 4) and $R_i\alpha_i > F(\rho)$ (because i is active), $U_i^*(\rho)$ is decreasing in $0 < \rho < \rho_i$. Furthermore, because $F'(\rho) > 0$, the effect of an increase in ρ on x_i^* is determined by whether i is in an aggressive or regressive position, i.e., by the sign of $R_i\alpha_i - 2F(\rho)$. Let $\hat{\rho}_i$ be the (unique) scalar satisfying $F(\hat{\rho}_i) = (R_i\alpha_i)/2$. When all other parameters are fixed, i is in an aggressive position if and only if $\rho < \hat{\rho}_i$ and in a regressive position if and only if $\hat{\rho}_i \leqslant \rho < \rho_i$. Furthermore, $x_i^*(\rho)$ is increasing for $\rho \in [0, \hat{\rho}_i]$, decreasing for $\rho \in [\hat{\rho}_i, \rho_i]$, and identically zero for $\rho \geqslant \rho_i$. The next example demonstrates the nonmonotonic behavior of $x_i^*(\rho)$.

EXAMPLE 6. Let $n = 2$, $\alpha_1 = R_1 = 1$, and $R_2\alpha_2 \leqslant 0.01$. For $\rho \geqslant 0.01$, firm 2 is inactive (because $\rho_2 < R_2\alpha_2 = 0.01 \leqslant \rho$) and $x_1^*(\rho) = (\sqrt{\rho} - \rho)^+$, which is not monotonic in ρ, e.g., $x_1^*(0.01) = x_1^*(0.81) = 0.09 < 0.16 = x_1^*(0.64)$. In this example, $\hat{\rho}_1 = 0.25 > 0$ and $\rho_1 = 1$, so $x_1^*(\rho)$ is increasing in $\rho \in [0, 0.25]$, decreasing in $\rho \in [0.25, 1]$, and is zero for $\rho \geqslant 1$.

Parameter $\alpha \equiv \alpha_j$ for $j \neq i$ where $R_j\alpha_j \geqslant F(z)$. By (A1) and (A2),

$$\frac{dU_i^*}{d\alpha}(\alpha) = \frac{-2F'(\alpha)}{\alpha_i}\left[1 - \frac{F(\alpha)}{R_i\alpha_i}\right] \quad \text{and}$$
$$\frac{dx_i^*}{d\alpha}(\alpha) = \frac{F'(\alpha)}{\alpha_i}\left[1 - \frac{2F(\alpha)}{R_i\alpha_i}\right].$$

As $F'(\alpha) > 0$ (by Lemma 4) and $R_i\alpha_i > F(\alpha)$ (because i is active), $U_i^*(\alpha)$ is decreasing in α. On the other hand, $x_i^*(\alpha)$ is increasing (decreasing) in α if i is in an aggressive (regressive) position. The next example demonstrates the nonmonotonic behavior of $x_i^*(\alpha_j)$.

EXAMPLE 7. Let $n = 2$, $R_1 = R_2 = 1$, $\alpha_1 = 10$, $\alpha_2 = \alpha$, and $\rho = 1$. Then $F(6) = 4.5705 < F(7.5) = 5.1244 < F(9) = 5.5850$ and $x_1^*(9) = 2.4658 < x_1^*(6) = 2.4816 < x_1^*(7.5) = 2.4985$.

Parameter $\alpha \equiv \alpha_i$. By (A1) and (A2),

$$\frac{dU_i^*}{d\alpha}(\alpha) = \frac{2(F'(\alpha)\alpha - F(\alpha))}{\alpha^2}\left[\frac{F(\alpha)}{R_i\alpha} - 1\right] \quad \text{and}$$
$$\frac{dx_i^*}{d\alpha}(\alpha) = \frac{\alpha F'(\alpha) - F(\alpha)}{\alpha^2}\left[1 - \frac{2F(\alpha)}{R_i\alpha}\right].$$

As $F(\alpha) - \alpha F'(\alpha) > 0$ (by Lemma 4) and $R_i\alpha_i > F(\alpha)$ (because i is active), U_i^* is increasing in α. Also, if i is in an aggressive (regressive) position, then x_i^* is decreasing (increasing) in α. The next example demonstrates the nonmonotonic behavior of $x_i^*(\alpha_i)$.

EXAMPLE 8. Let $n = R_1 = 1$, $\alpha_1 = \alpha$, and $\rho = 0.64$. Then $x_1^*(\alpha) = (0.8\sqrt{\alpha} - 0.64)/\alpha$; in particular, $x_1^*(1) = 0.16 = x_1^*(16)$ and $x_1^*(4) = 0.24$.

Table A.1. Sensitivity analysis of equilibrium expenditure and utility of firm i.

	ρ		$\alpha_j,\ j\neq i$ $[R_j\alpha_j \geqslant F(z)]$		α_i		$R_j,\ j\neq i$ $[R_j\alpha_j \geqslant F(z)]$		
	a	r	a	r	a	r	a	r	R_i
x_i^*	↑	↓	↑	↓	↓	↑	↑	↓	↑
U_i^*	↓		↓		↑		↓		↑

Revenue $R\equiv R_j$ of firm $j\neq i$ where $R_j\alpha_j \geqslant F(z)$. By (A1) and (A2), we imply that

$$\frac{dU_i^*}{dR}(R) = \frac{-2F'(R)}{\alpha_i}\left[1-\frac{F(R)}{R_i\alpha_i}\right] \quad \text{and}$$

$$\frac{dx_i^*}{dR}(R) = \frac{F'(R)}{\alpha_i}\left[1-\frac{2F(R)}{R_i\alpha_i}\right].$$

Therefore, the sensitivity results for R_j coincide with those for α_j with $j\neq i$.

Revenue $R\equiv R_i$ of firm i. By (A1),

$$\frac{dU_i^*}{dR}(R) = \left[1-\frac{F(R)}{R\alpha_i}\right]^2 + 2R\left[1-\frac{F(R)}{R\alpha_i}\right] \cdot \left[\frac{F(R)-F'(R)R}{R^2\alpha_i}\right] > 0,$$

the inequality following from $F(R)-RF'(R)>0$ (by Lemma 4) and $R_i\alpha_i > F(R)$ (because i is active). Thus, U_i^* is increasing in R. Also, (A2) implies that

$$\frac{dx_i^*}{dR}(R) = \frac{F'(R)}{\alpha_i}\left[1-\frac{F(R)}{R\alpha_i}\right] + \frac{F(R)}{(R\alpha_i)^2}[F(R)-RF'(R)] > 0,$$

the inequality following from $F'(R)>0$ and $F(R)-RF'(R)$ (by Lemma 4) and $R_i\alpha_i > F(R)$ (because i is active). Thus, $x_i^*(R_i)$ is increasing in R_i.

When "z" stands for "α_j" or "R_j" and $R_j\alpha_j < F(z)$, $x_i^*(z)$ and $U_i^*(z)$ are invariant of z and the corresponding sensitivity analysis is trivial. Table A.1 summarizes our findings about the sensitivity of the Nash equilibrium expenditures and utilities.

A.2. Globally Optimal Utility and Expenditure

By Theorem 3, the globally optimal solution $x^\star$ and its utility $U^\star \equiv U(x^\star)$ are zero if $\max_{i\in N} R_i\alpha_i \leqslant \rho$. Otherwise, with $i^\star \in \arg\max_{i\in N}(\sqrt{R_i}-\sqrt{\rho/\alpha_i})$, a globally optimal solution $x^\star$ is given by (27) and $U^\star = U(x^\star) = (\sqrt{R_{i^\star}}-\sqrt{\rho/\alpha_{i^\star}})^2$. Clearly, $U^\star$ is decreasing in ρ and (weakly) increasing in R_is and α_is. When a change does not affect $i^\star$, the sensitivity of $x_{i^\star}^\star$ to changes in ρ, $R_{i^\star}$ and $\alpha_{i^\star}$ is the same as the sensitivity of the equilibrium expenditure of a firm when it is the only active firm in the Nash equilibrium. An increase in ρ can eliminate a firm from consideration as a candidate for globally optimal monopoly. If R_j or α_j for some $j\neq i^\star$ increases, then j may replace $i^\star$ as the globally optimal monopoly, in which case $x_{i^\star}^\star$ becomes zero and $x_j^\star$ increases.

Appendix B. Efficiency

For systems such that $\max_{i\in N} R_i\alpha_i > \rho$, let the *efficiency index*, denoted by $\mathscr{E}$, be the ratio of the global utility of the Nash equilibrium to the optimal global utility; clearly, $0<\mathscr{E}\leqslant 1$. We recall that "the price of anarchy" is a commonly used measure of inefficiency for systems that entail costs rather than profits (see Koutsoupias and Papadimitriou 1999). The following example demonstrates that the system's efficiency index can be arbitrarily close to 0.

Example 4 (Continued). It was already observed that in Example 4, only firm 1 will be active in the (unique) Nash equilibrium and a monopolistic solution with firm 2 is the (unique) globally optimal solution. Theorem 2 and (19) imply that $F=\sqrt{R_1\alpha_1\rho} = 0.9220$ and the Nash equilibrium utility of firm 1, which equals the global utility, is $R_1(1-F/(R_1\alpha_1))^2 = (1-0.9220)^2 = 0.0061$. On the other hand, by Theorem 3, the optimal global utility is $(\sqrt{R_2}-\sqrt{\rho/\alpha_2})^2 = (\sqrt{10}-\sqrt{0.85/0.09})^2 = 0.0079$, so $\mathscr{E} = 0.0061/0.0079 = 0.7722$.

Suppose now for $M>1$, $R_2 = 10M$, $\alpha_2 = 0.09M^{-1}$, and all other parameters of Example 4 are unchanged. Because the $R_i\alpha_i$s do not change, the unique Nash equilibrium and its global utility (0.0061) will not change. Also, both firms continue to be candidates for the monopolistic firm in a globally optimal solution. Because $\sqrt{R_2}-\sqrt{\rho/\alpha_2} = 0.0890\sqrt{M} > 0.0780 = \sqrt{R_1}-\sqrt{\rho/\alpha_1}$, the monopolistic solution with firm 2 remains as the (unique) globally optimal solution. The optimal global utility is then $(\sqrt{R_2}-\sqrt{\rho/\alpha_2})^2 = M(\sqrt{10}-\sqrt{0.85/0.09})^2 = 0.0079M$, so $\mathscr{E} = 0.0061/(0.0079M) = 0.7722M^{-1}$, which can be arbitrarily small.

Example 4 shows how the system efficiency index diminishes when $R_i\alpha_i$s are kept the same while increasing the ratio of α_is. We next generalize the necessity of this construction by providing a lower bound on the system's efficiency index. Consider a system with $\max_{i\in N} R_i\alpha_i > \rho$ (else $\mathscr{E}$ is undefined) and $n>1$ (else $\mathscr{E} = 1$). Suppose $m\geqslant 1$ firms are active in the Nash equilibrium and firm j is the monopolistic firm in a globally optimal solution. Theorems 2 and 3 together with the fact that $R_1\alpha_1 \geqslant R_j\alpha_j > \rho$ for $j\leqslant m$ imply that

$$\begin{aligned}
\mathscr{E} &\equiv \sum_{i=1}^{m}\left(\frac{\sqrt{R_i}-F/(\alpha_i\sqrt{R_i})}{\sqrt{R_j}-\sqrt{\rho}/\sqrt{\alpha_j}}\right)^2 \\
&= \sum_{i=1}^{m}\frac{\alpha_j}{\alpha_i}\left(\frac{\sqrt{R_i\alpha_i}-F/\sqrt{R_i\alpha_i}}{\sqrt{R_j\alpha_j}-\sqrt{\rho}}\right)^2 \\
&\geqslant \sum_{i=1}^{m}\frac{\alpha_j}{\alpha_i}\left(\frac{\sqrt{R_i\alpha_i}-F/\sqrt{R_i\alpha_i}}{\sqrt{R_1\alpha_1}-\sqrt{\rho}}\right)^2.
\end{aligned}$$

If $m\geqslant 2$, then $R_2\alpha_2 > F$ and

$$\begin{aligned}
\mathscr{E} &\geqslant \frac{\alpha_j}{\alpha_1}\left(\frac{\sqrt{R_1\alpha_1}-F/\sqrt{R_1\alpha_1}}{\sqrt{R_1\alpha_1}-\sqrt{\rho}}\right)^2 \\
&= \frac{\alpha_j}{\alpha_1}\left(\frac{R_1\alpha_1 - F}{\sqrt{R_1\alpha_1}(\sqrt{R_1\alpha_1}-\sqrt{\rho})}\right)^2 \\
&\geqslant \frac{\alpha_j}{\alpha_1}\left(\frac{R_1\alpha_1 - R_2\alpha_2}{R_1\alpha_1}\right)^2 \geqslant \left(\min_{k\in N}\frac{\alpha_k}{\alpha_1}\right)\left(1-\frac{R_2\alpha_2}{R_1\alpha_1}\right)^2.
\end{aligned}$$

Alternatively, if $m=1$, then $F=\sqrt{\rho R_1\alpha_1}$ and

$$\begin{aligned}
\mathscr{E} &= \frac{\alpha_j}{\alpha_1}\left(\frac{\sqrt{R_1\alpha_1}-\sqrt{\rho}}{\sqrt{R_j\alpha_j}-\sqrt{\rho}}\right)^2 \geqslant \frac{\alpha_j}{\alpha_1} \\
&\geqslant \min_{k\in N}\frac{\alpha_k}{\alpha_1} \geqslant \left(\min_{k\in N}\frac{\alpha_k}{\alpha_1}\right)\left(1-\frac{R_2\alpha_2}{R_1\alpha_1}\right)^2.
\end{aligned}$$

Endnotes

1. Members of the exponential family are reasonable approximations to reality when there is a need to assess achievement in R&D activities where success is rare, where promising ideas or designs often turn out to be infeasible, and where higher levels of success are increasingly unlikely; see the discussion in Gerchak and Kilgour (1999). The assumption that investment levels linearly affect the exponential rates is quite common in similar studies where it can be regarded as a first-order approximation of reality.
2. The computation of the ρ_js avoids the need to evaluate each $F_j(\rho)$ and find the interval $[R_{i+1}\alpha_{i+1}, R_i\alpha_i]$ in which it lies. This procedure requires taking n square roots (when $\rho > 0$).
3. Consider firm i and $\theta > 1/\alpha_i$. If firm i invests $\epsilon > 0$ and the total investment of the other firms is $\sum_{k\in N\setminus\{i\}} \alpha_k x_k = (\theta\alpha_i - 1)\epsilon$, then $U_i(x) = \theta - \epsilon$ and the limit of $U_i(\cdot)$ along a trajectory approaching 0 depends on the selected trajectory.

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