J Optim Theory Appl (2012) 154:986–1014
DOI 10.1007/s10957-012-0066-x

A Stochastic Competitive Research and Development Race Where "Winner Takes All" with Lower and Upper Bounds

Pelin G. Canbolat · Boaz Golany · Uriel G. Rothblum

Received: 28 December 2011 / Accepted: 21 April 2012 / Published online: 28 July 2012

Abstract The paper analyzes an environment in which several firms compete over the development of a project. Each firm decides how much to invest in the project while adhering to firm-specific lower and upper investment bounds. The completion time of the project by a firm has exponential distribution with rate that depends linearly on the investment of the firm. The firm that completes the project first collects all its revenues whereas the remaining firms earn nothing. The paper establishes the existence and uniqueness of both the Nash equilibrium and the globally optimal solution, provides explicit representations parametrically in the interest rate, and constructs computationally efficient methods to solve these two problems. It also examines sensitivity of Nash equilibrium to marginal changes in lower and upper bounds.

Keywords R&D management · Nash equilibria · Global optimality · Resource allocation

1 Introduction

Recently, Canbolat et al. [1] proposed an efficient approach to compute the (unique) Nash equilibrium in Research and Development (R&D) competitions over the devel-

---

Communicated by Boris Mordukhovich.

U.G. Rothblum deceased on March 12, 2012.

P.G. Canbolat · B. Golany (✉) · U.G. Rothblum
Faculty of Industrial Engineering and Management, The Technion, Israel Institute of Technology, Haifa, Israel
e-mail: golany@ie.technion.ac.il

P.G. Canbolat
e-mail: canbolat.pg@gmail.com

opment of a single technology-intensive product or process, where each firm has to decide whether to enter the race and if so, how much to invest in it. They characterized the competition as a *winner-takes-all* mechanism, as the revenues are collected only by the firm that completes the project first while all other firms earn nothing. In this model, the development time of the project by each firm follows an exponential distribution with rate proportional to its investment and the (random) completion times are independent across firms. The winner collects the revenues upon the successful completion of the project, costs and revenues are continuously discounted.

This paper extends their results to the case where the amounts that the firms can invest are subject to lower and upper bound constraints. In particular, it provides explicit representations of a unique Nash equilibrium and a globally optimal solution as functions of the system parameters, including the interest rate and the bounds. With the introduction of lower and upper bound constraints, the computation of the unique Nash equilibrium involves computing four *breakpoints* for each firm. The equilibrium investment of a firm depends on the comparison of the interest rate with these breakpoints. Specifically, the four breakpoints define five intervals that determine whether the firm invests its lower bound (the minimum amount it must invest), its upper bound (its full budget), or some amount in between these bounds. The analysis in this paper provides closed-form expressions for the firms' equilibrium investments and utilities, which enables the discussion of the sensitivity of the equilibrium outcome to changes in the upper or lower bound of a specific firm. The paper then proceeds by introducing a centralized decision-making version of the model with the objective of maximizing the global utility (i.e., the sum of the firms' utilities) without violating their individual constraints. After identifying the structure of the optimal solution, it provides an explicit formula for it. In particular, the results obtained here indicate that the globally optimal solution exhibits concentration of effort in a *greedy* way (to the degree that the individual constraints permit) whereas diversification is commonly observed in a Nash equilibrium when the parameters of the firms do not exhibit much variability.

Upper bounds on investments express limited availability of resources, a common restriction in corporate and personal decision making. Lower bounds on investments can model commitments that firms have already made. In particular, an analysis that allows for lower bounds can be the basis of a dynamic model where investment decisions are reviewed in view of new information, but past commitments cannot be voided.

The problem of selecting the R&D projects to invest in and determining the amount of investment in each of the selected projects is common in many markets, especially in markets where the speed of innovation matters. This problem has been popular in both operations research and economics literature (see [1] for a review of these); however, in the game theory literature, the focus has traditionally been on establishing existence and uniqueness of Nash equilibrium rather than computing it explicitly. The current paper obtains existence and uniqueness results for a Nash equilibrium and a globally optimal solution by explicit construction.

As the model of Gerchak and Parlar [2], the model in this paper includes budget constraints. But unlike their model, it considers a single R&D race with exponential completion times under continuous discounting. Further, revenues from the project are allowed to vary among firms. As for analysis, [2] numerically compute a Nash equilibrium only for instances with two firms, two projects and exponential distributions. Also, neither [2] nor other references in [1] address the problem of global optimization; this paper also solves that problem and compares the structure of its solution to the structure of the Nash equilibrium.

The paper is organized as follows. Section 2 introduces the formal model and the notation used in the paper. Section 3 constructs a unique Nash equilibrium solution by providing an explicit formula and develops an efficient method to compute it. Section 4 examines marginal effects of lower and upper bound constraints on the Nash equilibrium solution. Section 5 considers a centralized version of the problem, solves for a globally optimal solution in closed form, and discusses its efficient computation. Section 6 provides some extensions of the model, and Sect. 7 concludes with a summary. All the proofs are in the Appendix.

2 The Formal Model

The model considered in this paper builds on the one formulated and studied in [1], incorporating bounds on investment levels. Specifically, there are n firms competing in an R&D race. The decision variable of firm $i \in N := \{1, \ldots, n\}$ is the amount x_i (in \$) which it allocates to funding the project. This is a one-time payment made at the beginning of the horizon and cannot be retrieved regardless of the actual length of the project (e.g., equipment purchase, outsourcing contracts, etc.). The (random) completion time of the project by firm i, denoted T^i, follows an exponential distribution whose rate depends linearly on the funding level x_i. Specifically, if firm i allocates x_i to the project, then the rate of the exponential distribution of T^i is $\alpha_i x_i$, where $\alpha_1, \ldots, \alpha_n$ are given positive constants (in $\frac{1}{\$\times\text{time-unit}}$). The T^i's are stochastically independent over firms (under all potential joint funding levels). The evolution of time is continuous and a fixed interest rate $\rho > 0$ is applied to costs and revenues, i.e., a reward received at time t is discounted by $e^{-\rho t}$. When firm i is the first one to complete the project, it earns, upon completion, the revenue $R_i > 0$ whereas all other firms get nothing (and lose their investment). This situation is referred to as "winner-takes-all". The new aspect in the current paper is the introduction of lower and upper bound constraints on each firm's investment. The investment x_i of firm i must satisfy the constraints $0 \leq L_i \leq x_i \leq B_i \leq \infty$. Hence, the parameters of the problem are the α_i's, the R_i's, the L_i's, the B_i's, and ρ.

Let $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ be the *(joint) investment vector* and for $i \in N$, write $x = (x_i, x_{-i})$ with $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. For $x \in \mathbb{R}^n$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$, their scalar product is denoted as $\alpha^T x$ $(= \sum_{k=1}^n \alpha_k x_k)$. Using the fact that T^i's are independent and exponentially distributed, [1] proved that the utility (i.e., the expected net profit) of firm i can be expressed as a function of the joint investment vector $x \geq 0$ by

$$U_i(x) = \frac{R_i \alpha_i x_i}{\alpha^T x + \rho} - x_i \tag{1}$$

and that $U_i(x)$ is concave in $x_i \geq 0$ for each $i \in N$.

A Nash equilibrium is a vector $x^* \in \mathbb{R}^n$ satisfying

$$U_i\big(x^*\big) = \max_{L_i \le x_i \le B_i} U_i\big(x_i, x^*_{-i}\big) \quad \text{for each } i \in N. \tag{2}$$

3 Explicit Representation of a Unique Nash Equilibrium

This section derives an explicit solution for a unique Nash equilibrium, parameterized by the interest rate ρ. [1] analyzed the model where firms are unconstrained in the nonnegative amounts that they can invest in R&D projects, i.e., $L_i = 0$ and $B_i = \infty$ for each $i \in N$. Henceforth, refer to that model as the *relaxed* model. The development herein focuses on the model with bounds on investment levels and its new features. For $x \in \mathbb{R}^n$ with $x_i > \max\{L_i, R_i\}$, $U_i(x_i, x_{-i}) < U_i(\max\{L_i, R_i\}, x_{-i})$, implying that every Nash equilibrium x^* for the relaxed model satisfies $x^*_i \le \max\{L_i, R_i\}$, so $B_i = \max\{L_i, R_i\}$ is equivalent to $B_i \in [\max\{L_i, R_i\}, \infty]$.[1] Also, if $B_i = 0$, firm i's investment can be set to zero and firm i can be eliminated from the model. Therefore, the rest of the paper assumes without loss of generality that $0 < B_i \le \max\{L_i, R_i\}$ for each $i \in N$.

When the investment of firm i is restricted to be in the interval $[L_i, B_i]$ with $0 \le L_i \le B_i \le \infty$, (2) implies that a vector $x^* \in \mathbb{R}^n$ is a Nash equilibrium if and only if

$$x^*_i \in \operatorname*{arg\,max}_{L_i \le x_i \le B_i} U_i\big(x_i, x^*_{-i}\big) \quad \text{for } i \in N. \tag{3}$$

As each U_i is concave in x_i and each x_i is essentially restricted to the compact convex interval $[L_i, \max\{L_i, R_i\}]$, the classical result of Rosen [3] guarantees the existence of a Nash equilibrium.[2] Given x_{-i}, the problem of firm i is to maximize its concave utility U_i subject to lower and upper bounds that conform to constraint qualification; therefore, for each i, Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient to ensure (3). Hence, x^* is a Nash equilibrium if and only if it satisfies the following KKT conditions:

$$\frac{R_i\alpha_i(\alpha^T x^* - \alpha_i x^*_i + \rho)}{(\alpha^T x^* + \rho)^2} + \tau_i - \sigma_i = 1, \tag{4}$$

$$L_i \le x^*_i \le B_i, \tag{5}$$

$$\tau_i \ge 0, \sigma_i \ge 0, \tag{6}$$

$$\tau_i\big(x^*_i - L_i\big) = 0, \quad \text{and} \tag{7}$$

$$\sigma_i\big(B_i - x^*_i\big) = 0. \tag{8}$$

[1] In fact, [1] showed that when $B_i = \infty$ for $i \in N$, in the unique Nash equilibrium, the investment of each firm i is always bounded by $R_i/4$. Consequently, if the budget B_i of firm i exceeds $R_i/4$ for every $i \in N$, then the unique Nash equilibrium of the relaxed model is feasible for the model with the budget constraints and is therefore also a Nash equilibrium for the latter.

[2] Uniqueness also follows from [3]; however, this paper develops a constructive argument that provides an efficient method to compute the Nash equilibrium in addition to proving its existence and uniqueness.

The following theorem is essential to the characterization of the set X^* of all Nash equilibria.

Theorem 3.1 *Let $\mathcal{Q}$ be the set of quadruples (F, J, K, M) where*

(a) *F is the unique positive solution of the equation*

$$\left(\sum_{i\in M}\frac{1}{R_i\alpha_i}\right)z^2-\big(|M|-1\big)z-\left(\sum_{i\in J}B_i\alpha_i+\sum_{i\in K}L_i\alpha_i+\rho\right)=0 \tag{9}$$

and

(b)

$$J=\left\{i\in N:\frac{F(R_i\alpha_i-F)}{R_i\alpha_i^2}\geq B_i\right\},$$

$$K=\left\{i\in N:\frac{F(R_i\alpha_i-F)}{R_i\alpha_i^2}\leq L_i\right\},$$

$$M=\left\{i\in N:L_i<\frac{F(R_i\alpha_i-F)}{R_i\alpha_i^2}<B_i\right\}.$$

Then the correspondence that maps $(F, J, K, M)\in\mathcal{Q}$ into $x^*\in\mathbb{R}^n$ with

$$x_i^*=\begin{cases}B_i, & \text{if } i\in J,\\ L_i, & \text{if } i\in K,\\ \frac{F(R_i\alpha_i-F)}{R_i\alpha_i^2}, & \text{if } i\in M,\end{cases} \tag{10}$$

is one-to-one from $\mathcal{Q}$ onto X^*. Further, the inverse correspondence that maps $x^*\in X^*$ into $(F, J, K, M)\in\mathcal{Q}$ has

$$F=\alpha^T x^*+\rho, \tag{11}$$

$$J=\{i\in N:x_i^*=B_i\}, \tag{12}$$

$$K=\{i\in N:x_i^*=L_i\},\quad \textit{and} \tag{13}$$

$$M=\{i\in N:L_i<x_i^*<B_i\}. \tag{14}$$

Henceforth, the paper refers to Conditions (a)–(b) without explicit reference to Theorem 3.1.

Theorem 3.1 shows that the search for Nash equilibria reduces to a search for quadruples (F, J, K, M) satisfying Conditions (a)–(b); the corresponding Nash equilibrium can then be constructed by (10). Evidently, given the sets (J, K, M), (a) uniquely determines the scalar F. Conversely, given a scalar F, (b) uniquely determines the sets (J, K, M). Hence, satisfying both (a) and (b) means that F and (J, K, M) are determined, respectively, from each other.

The next corollary of Theorem 3.1 expresses a Nash equilibrium as a function of F, suppressing the dependence on (J, K, M).

Corollary 3.1 *If x^* is a Nash equilibrium corresponding to (F, J, K, M), then*

$$x_i^* = \begin{cases} L_i & \text{if } \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} \le L_i, \\ \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} & \text{if } L_i < \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} < B_i, \\ B_i & \text{if } \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} \ge B_i. \end{cases} \tag{15}$$

Theorem 3.1 also implies the following relationships between firms' equilibrium investments.

Corollary 3.2 *Let x^* be a Nash equilibrium.*

(i) *If $x_i^* > L_i$, $R_j\alpha_j > R_i\alpha_i$ and $L_j = 0$, then $x_j^* > L_j$.*
(ii) *If $x_i^* > L_i$, $R_j\alpha_j > R_i\alpha_i$ and $B_j\alpha_j > B_i\alpha_i$, then $\alpha_j x_j^* > \alpha_i x_i^*$.*
(iii) *If $x_i^* > L_i \ge L_j$, $R_j\alpha_j > R_i\alpha_i$ and $\alpha_j < \alpha_i$, then $x_j^* > L_j$.*

The straightforward approach to compute a Nash equilibrium, as implied by Theorem 3.1, is to solve (9) for any partitioning (J, K, M) and test if the solution obtained satisfies the conditions of Theorem 3.1. The rest of this section first partitions the domain of F into intervals each of which corresponds to a unique partitioning (J, K, M) of firms. In addition to reducing the set of feasible partitions to test, this paves the way for parameterizing the model in the interest rate. Then for any one of these intervals, the paper expresses F as a function of the interest rate ρ and combines these functions together to represent F as a continuous and increasing function of ρ. Each boundary value for F corresponds to a boundary value for ρ, so the parameterization in ρ allows translating the intervals of F into intervals of ρ. The comparison of the given interest rate to these boundaries, in turn, uniquely determines the Nash equilibrium.

Corollary 3.1 established the representation of Nash equilibrium investment of each firm in terms of $F \ge 0$ through the function (15), illustrated in Fig. 1. It implies

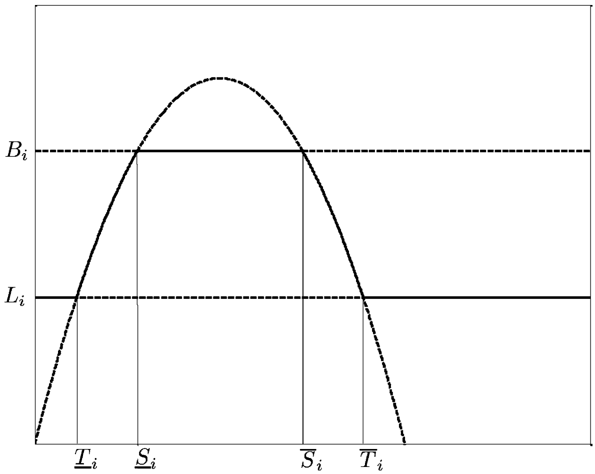

Fig. 1 x_i^* as a function of F

that for each firm i, the positive real line can be partitioned into intervals such that the interval including F determines which of the sets (J, K, M) firm i belongs to. Determining the boundaries of these intervals requires computing the values at which either the lower or the upper bound of firm i is binding. By Corollary 3.1, if x^* is a Nash equilibrium corresponding to $(F, J, K, M) \in \mathcal{Q}$, then the upper bound of firm i is binding if and only if $F^2 - R_i\alpha_i F + R_i\alpha_i^2 B_i \leq 0$. This inequality holds for no $F \in \mathbb{R}$ if $B_i > R_i/4$ and it holds for $F \in [\underline{S}_i, \overline{S}_i]$ if $B_i \leq R_i/4$, where

$$\underline{S}_i := \frac{R_i\alpha_i}{2}\left(1 - \sqrt{1 - \frac{4B_i}{R_i}}\right) \quad \text{and} \quad \overline{S}_i := \frac{R_i\alpha_i}{2}\left(1 + \sqrt{1 - \frac{4B_i}{R_i}}\right). \tag{16}$$

Similarly, the lower bound of firm i is binding if and only if $F^2 - R_i\alpha_i F + R_i\alpha_i^2 L_i \geq 0$. This inequality holds for all $F \in \mathbb{R}$ if $L_i \geq R_i/4$ and for $F \in]-\infty, \underline{T}_i] \cup [\overline{T}_i, +\infty[$ if $L_i < R_i/4$, where

$$\underline{T}_i := \frac{R_i\alpha_i}{2}\left(1 - \sqrt{1 - \frac{4L_i}{R_i}}\right) \quad \text{and} \quad \overline{T}_i := \frac{R_i\alpha_i}{2}\left(1 + \sqrt{1 - \frac{4L_i}{R_i}}\right). \tag{17}$$

So, if $\frac{R_i}{4} \leq L_i < B_i$, then the lower bound of firm i is always binding and its upper bound is never binding. Henceforth, assume that $L_i < \frac{R_i}{4}$ for each $i \in N$, as the extension of forthcoming results to the case where this assumption is relaxed is straightforward. Then three cases arise:

(1) If $0 \leq L_i < B_i < \frac{R_i}{4}$, then $0 \leq \underline{T}_i < \underline{S}_i < \frac{R_i\alpha_i}{2} < \overline{S}_i < \overline{T}_i \leq R_i\alpha_i$.
(2) If $0 \leq L_i < \frac{R_i}{4} = B_i$, then $0 \leq \underline{T}_i < \underline{S}_i = \frac{R_i\alpha_i}{2} = \overline{S}_i < \overline{T}_i \leq R_i\alpha_i$.
(3) If $0 \leq L_i < \frac{R_i}{4} < B_i$, then $\underline{S}_i$ and $\overline{S}_i$ are not defined and $0 \leq \underline{T}_i < \frac{R_i\alpha_i}{2} < \overline{T}_i \leq R_i\alpha_i$.

Merging all these individual partitions yields a finer partition of the positive real line such that each one of the finitely many intervals corresponds to a unique partition (J, K, M) of firms. Merging requires ordering the boundaries for all the firms. To do this rigorously, let

$$\Phi := \left[\cup\left\{\{\overline{S}_i, \underline{S}_i\} : i \in N \text{ with } B_i \leq R_i/4\right\}\right] \cup \left[\cup\left\{\{\overline{T}_i, \underline{T}_i\} : i \in N\right\}\right]. \tag{18}$$

If $L_i = 0$ for $i \in N$, then $\underline{T}_i = 0$, so $|\{i \in N : L_i = 0\}|$ of the elements generating Φ are zeros. If for all $i \in N$, $L_i = 0$ and $B_i \geq R_i/4$, then $\Phi = \cup\{\{\overline{T}_i, \underline{T}_i\} : i \in N\} = \{R_i\alpha_i : i \in N\} \cup \{0\}$, consistently with the results of [1]. To avoid discussion of degenerate cases, assume that the positive elements generating Φ are distinct. This assumption excludes the case with $\overline{S}_i = \underline{S}_i$ for some $i \in N$, which occurs if and only if $B_i = R_i/4$. In this case, $x_i^* = B_i$ if and only if $x_i^* = \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2}$. With (15), this implies that the budget constraint of firm i is always redundant even though it can be binding in the marginal case when its unconstrained equilibrium investment equals the budget. While multiple occurrence of zero among the elements generating Φ is allowed, in such cases, they are accounted as one element in Φ. Under the distinctness assumption, (18) holds with $B_i < R_i/4$ replacing $B_i \leq R_i/4$. Next, order the elements of Φ in decreasing order, so, $\varphi_1 := \max \Phi$ and $\varphi_t := \max(\Phi \setminus \{\varphi_1, \ldots, \varphi_{t-1}\})$ for $2 \leq t \leq |\Phi|$. Let $\varphi_0 := \infty$.

For $t = 0, 1, \ldots, |\Phi|$, then define

$$J_t := \{i \in N : \underline{S}_i < \varphi_t \le \overline{S}_i \text{ and } B_i < R_i/4\}, \tag{19}$$

$$K_t := \{i \in N : \varphi_t \le \underline{T}_i \text{ or } \varphi_t > \overline{T}_i\}, \tag{20}$$

$$M_t := N \setminus (J_t \cup K_t), \tag{21}$$

$$\gamma_t := \sum_{i \in M_t} \frac{1}{R_i \alpha_i}, \qquad \theta_t := \sum_{i \in J_t} B_i \alpha_i, \qquad \zeta_t := \sum_{i \in K_t} L_i \alpha_i \quad \text{and} \quad v_t := |M_t| - 1; \tag{22}$$

in particular, $J_0 = M_0 = \emptyset$, $K_0 = N$, $\gamma_0 = \theta_0 = 0$, and $\zeta_0 = \sum_{i \in N} L_i \alpha_i$. Also, $v_t \ge 0$ if and only if $M_t \ne \emptyset$ and $v_t = -1$ if and only if $M_t = \emptyset$ (in particular, $v_0 = -1$). Also, with $\mathbb{R}_+$ as the set of positive real numbers, let $F_t : \mathbb{R}_+ \to \mathbb{R}$ where for $\rho > 0$

$$F_t(\rho) = \begin{cases} \frac{v_t + \sqrt{v_t^2 + 4\gamma_t(\theta_t + \zeta_t + \rho)}}{2\gamma_t}, & \text{if } \gamma_t > 0 \text{ (i.e., } v_t \ge 0), \\ \theta_t + \zeta_t + \rho, & \text{if } \gamma_t = 0 \text{ (i.e., } v_t = -1). \end{cases} \tag{23}$$

The following constants turn out to be central to the efficient computation of a Nash equilibrium for the model considered in this paper:

$$\rho_t := \begin{cases} \infty, & \text{if } t = 0, \\ \gamma_t \varphi_t^2 - v_t \varphi_t - \theta_t - \zeta_t, & \text{if } t \in \{1, \ldots, |\Phi|\}, \\ -\infty, & \text{if } t = |\Phi| + 1. \end{cases} \tag{24}$$

In particular, $\varphi_1 = \overline{T}_i$ for some $i \in N$ and $\rho_1 = \overline{T}_i - \sum_{k \in N} L_k \alpha_k$. Then either $\varphi_2 = \overline{T}_j$ for some $j \ne i$ or $\varphi_2 \in \{\underline{T}_i, \overline{S}_i\}$. If $\varphi_2 = \overline{T}_j$ for some $j \in N \setminus \{i\}$, then $\rho_2 = \frac{\overline{T}_j^2}{R_i \alpha_i} - \sum_{k \in N \setminus \{i\}} L_k \alpha_k$. If $\varphi_2 = \underline{T}_i$, then $\rho_2 = \underline{T}_i - \sum_{k \in N} L_k \alpha_k$. If $\varphi_2 = \overline{S}_i$, then $\rho_2 = \overline{S}_i - B_i \alpha_i - \sum_{k \in N \setminus \{i\}} L_k \alpha_k$.

The computation of the ρ_t's is readily available from (24), as demonstrated in the next example.

Example 3.1 Table 1 exhibits the parameters α_i and R_i for an instance with three firms and computes the elements of Φ.

The values $\overline{S}_3$ and $\underline{S}_3$ are not defined, as $B_3 = 1 > R_3/4 = 0.5$. Consequently, $\varphi_1 = 10$, $\varphi_2 = 7.2361$, $\varphi_3 = 5$, $\varphi_4 = 3.6180$, $\varphi_5 = 2.7639$, $\varphi_6 = 2$ and $\varphi_7 = 1.3820$. Corresponding ρ_t's are: $\rho_1 = \frac{1}{10} 10^2 = 10$, $\rho_2 = 7.2361 - 2 = 5.2361$, $\rho_3 = \frac{1}{5} 5^2 - 2 = 3$, $\rho_4 = 3.6180 - 2 - 1 = 0.6180$, $\rho_5 = \frac{1}{10} 2.7639^2 - 1 = -0.2361$, $\rho_6 = (\frac{1}{10} + \frac{1}{2}) 2^2 - 2 - 1 = -0.6$ and $\rho_7 = (\frac{1}{10} + \frac{1}{5} + \frac{1}{2}) 1.3820^2 - 2(1.3820) = -1.2361$.

Table 1
Example 3.1—Parameters

i	R_i	α_i	L_i	B_i	$\overline{T}_i = R_i \alpha_i$	$\underline{T}_i$	$\overline{S}_i$	$\underline{S}_i$
1	10	1	0	2	10	0	7.2361	2.7639
2	5	1	0	1	5	0	3.6180	1.3820
3	2	1	0	1	2	0	–	–

Fig. 2 Lemma 3.1

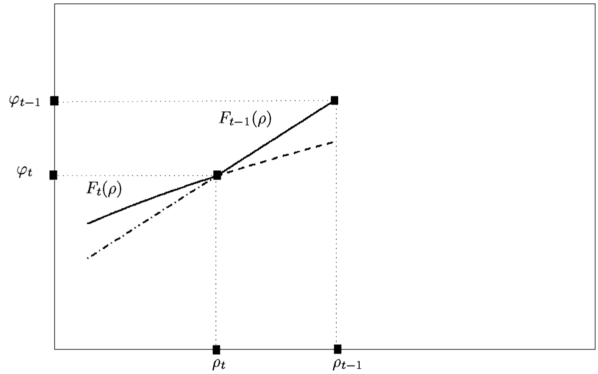

Table 2 Example 3.1—$F_t(\cdot)$'s

t	0	1	2	3	4	5	6	7
$F_t(\rho)$	ρ	$\sqrt{10\rho}$	$2+\rho$	$\frac{\sqrt{0.8(2+\rho)}}{0.4}$	$3+\rho$	$\frac{\sqrt{0.4(1+\rho)}}{0.2}$	$\frac{1+\sqrt{1+2.4(1+\rho)}}{1.2}$	$\frac{2+\sqrt{4+3.2\rho}}{1.6}$

If $\gamma_t > 0$, the first line of (23) extends $F_t(\rho)$ to $\rho \le 0$ satisfying $v_t^2 + 4\gamma_t(\theta_t + \zeta_t + \rho) \ge 0$. If $\gamma_t = 0$, then the second line of (23) extends $F_t(\rho)$ to any $\rho \le 0$. Clearly, $F_t(\cdot)$'s are strictly increasing on their extended domains. The next result establishes, among other facts, that the joint ranking of the φ_t's and ρ_t's in Example 3.1 is not a coincidence.

Lemma 3.1

(i) *For $t = 1, \dots, |\Phi|$, ρ_t is in the extended domain of $F_t(\cdot)$ and $F_{t-1}(\cdot)$, and $F_t(\rho_t) = \varphi_t = F_{t-1}(\rho_t)$ (see Fig. 2).*
(ii) $-\infty = \rho_{|\Phi|+1} < \rho_{|\Phi|} < \dots < \rho_1 < \rho_0 = \infty$.

Example 3.1 (continued) In Example 3.1, $\gamma_1 = \gamma_5 = 0.1$, $\gamma_2 = \gamma_4 = 0$, $\gamma_3 = 0.2$, $\gamma_6 = 0.6$, and $\gamma_7 = 1.1$. Table 2 provides the corresponding expressions for the $F_t(\cdot)$'s and Fig. 3 exhibits these increasing (concave) functions, with ρ_t's marked on the x axis and φ_t's marked on the y axis.

The following theorem is the main result of this section.

Theorem 3.2 *Suppose $t = 0, \dots, |\Phi|$ is the unique integer with $\rho_{t+1} \le \rho < \rho_t$ and $\mathcal{Q}$ is defined as in Theorem 3.1. Then:*

(i) $\mathcal{Q} = \{(F_t(\rho), J_t, K_t, M_t)\}$.
(ii) *x^* given by (10) with $(F, J, K, M) = (F_t(\rho), J_t, K_t, M_t)$ is a unique Nash equilibrium and*

$$U_i(x^*) = \begin{cases} B_i(\frac{R_i\alpha_i}{F} - 1), & \text{if } i \in J, \\ L_i(\frac{R_i\alpha_i}{F} - 1), & \text{if } i \in K, \\ R_i(1 - \frac{F}{R_i\alpha_i})^2, & \text{otherwise.} \end{cases} \tag{25}$$

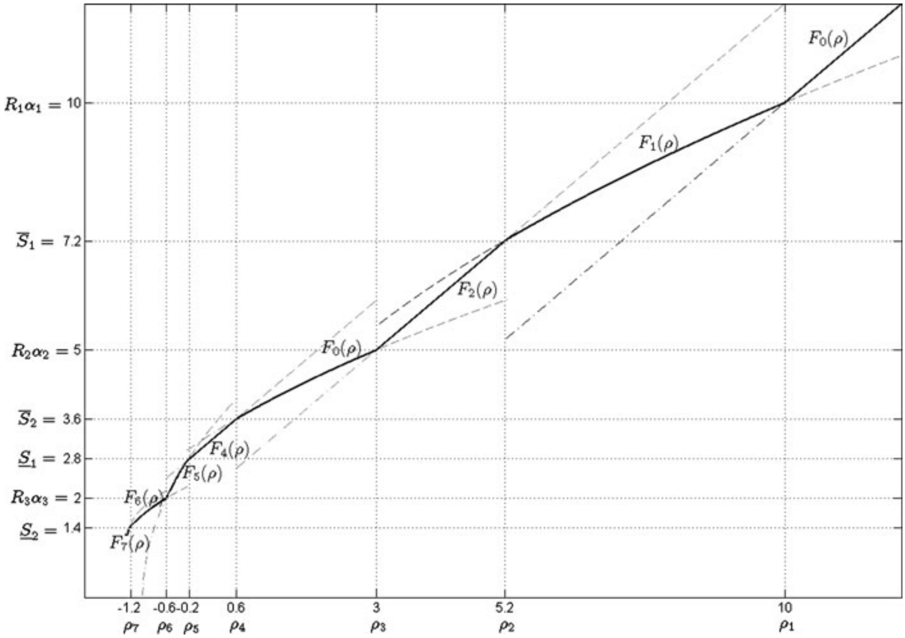

Fig. 3 $F_m(\rho)$ in Example 3.1

To compute the unique Nash equilibrium explicitly, consider Example 3.1 and Fig. 3. Suppose the interest rate is $\rho = 0.5$, so $\rho_5 < \rho < \rho_4$. Since $\varphi_4 = 3.6180$, $J_4 = \{i \in N : \underline{S}_i < 3.6180 \le \overline{S}_i\} = \{1, 2\}$, $K_4 = \{i \in N : 3.6180 \le \underline{T}_i \text{ or } 3.6180 > \overline{T}_i\} = \{3\}$ and $M_4 = N \setminus (J_4 \cup K_4) = \emptyset$ satisfy the assumption of Theorem 3.2. As determined from Fig. 3, the value F corresponding to the unique Nash equilibrium (by (10)) is then $F = F(\rho) = F_4(\rho) = 3.5$.

Computation of Nash Equilibrium Theorems 3.1 and 3.2 indicate that the (unique) Nash equilibrium investments and utilities can be easily computed from corresponding values t and F. A geometric approach for determining t and F using the graph that plots $F(\rho)$ vs. ρ was just illustrated for Example 3.1 (with Fig. 3). The following is the algebraic method.

(i) For $i = 1, \ldots, n$, calculate $(\underline{T}_i, \overline{T}_i)$ using (17), calculate $(\underline{S}_i, \overline{S}_i)$ using (16) if $B_i < R_i/4$ and let Φ be the set of generated elements.
(ii) Order the elements of Φ in decreasing order, say $\varphi_1 > \varphi_2 > \cdots > \varphi_{|\Phi|}$.
(iii) For $j = 1, \ldots, |\Phi| + 1$, compute $J_j, K_j, M_j, \gamma_j, \theta_j, \zeta_j, \upsilon_j$ and ρ_j using (19)–(22) and (24) and test if $\rho_j \le \rho$. For the first j such that $\rho_j \le \rho$, set $t = j - 1$.
(iv) Compute $F_t(\rho)$ using (23). For $i \in N$, compute x_i^* by letting $(F, J, K, M) = (F_t(\rho), J_t, K_t, M_t)$ in (10) and compute U_i^* from (25). Output (x_i^*, U_i^*) for every $i \in N$.

Evidently, t found in (iii) satisfies the assumptions of Theorem 3.2 and, therefore, the conclusions of this theorem assure that x^* is the unique Nash equilibrium and $U_i^* = U_i(x^*)$ for each $i \in N$. The computation of these values involves $O(n)$ arithmetic operations and up to n square roots in step (i), $O(n \log n)$ comparisons in step (ii), $O(n^2)$ comparisons and arithmetic operations in step (iii), a single square root and $O(1)$ arithmetic operations in step (iv). Hence, the overall computational effort of finding the unique Nash equilibrium with this algorithm is $O(n^2)$.

4 Sensitivity of Nash Equilibrium to Lower and Upper Bounds

Sensitivity analysis with respect to lower and upper bounds follows the same lines. For this reason, this section conducts sensitivity analysis with respect to upper bounds in detail first and then summarizes how to adapt this analysis to lower bounds.

4.1 Marginal Effect of Upper Bounds

This section studies the effects of tightening the upper bound constraint of a firm on the Nash equilibrium investments x^* and utilities $U_i^* := U_i(x^*)$. Henceforth, a firm is referred to as *active* if it invests a positive amount and the terms "increasing/decreasing" are used in the weak sense.

To start the sensitivity analysis, express all system variables as a function of $B := (B_1, \dots, B_n)$, i.e., write $x_i^*(B)$, $U_i^*(B)$, $F(B)$, $J(B)$, $K(B)$, and $M(B)$. By (10) in Theorem 3.1,

$$x_j^*(B) = \begin{cases} B_j, & \text{if } j \in J(B), \\ L_j, & \text{if } j \in K(B), \\ \frac{F(B)(R_j\alpha_j - F(B))}{R_j\alpha_j^2}, & \text{if } j \in M(B) = N \setminus (J(B) \cup K(B)), \end{cases} \tag{26}$$

and by (25) in Theorem 3.2,

$$U_j^*(B) = \begin{cases} B_j(\frac{R_j\alpha_j}{F(B)} - 1), & \text{if } j \in J(B), \\ L_j(\frac{R_j\alpha_j}{F(B)} - 1), & \text{if } j \in K(B), \\ R_j(1 - \frac{F(B)}{R_j\alpha_j})^2, & \text{if } j \in M(B) = N \setminus (J(B) \cup K(B)). \end{cases} \tag{27}$$

4.1.1 Effect on Equilibrium Activity in Market

From (16), it follows that if the upper bound B_i of firm i decreases, then $\underline{S}_i$ decreases and $\overline{S}_i$ increases. By (23), $F_t(\rho)$ is increasing in $\rho > 0$, so ρ_t that uniquely solves the equation $F_t(\rho) = \underline{S}_i$ decreases and $\rho_{t'}$ that uniquely solves the equation $F_{t'}(\rho) = \overline{S}_i$ increases. Consequently, the interval of the interest rates for which the upper bound constraint of firm i is binding expands. This implies that a decrease in the upper bound of an active firm cannot drive it out of the competition even though it can reduce its equilibrium investment.

On the other hand, ρ_s does not change for s such that $\varphi_s > \overline{S}_i$. If $\varphi_s < \overline{S}_i$, then ρ_s increases as the bound B_i decreases; therefore, for a given interest rate $\rho > 0$, a firm j that is active when the bound of firm i is B_i will be active at all smaller upper bounds of firm i. In other words, firm j becomes more tolerant to discounting and invests positive amounts for higher interest rates as the budget of firm i is reduced. This suggests that restricting the investment of the *stronger* firms, i.e., the firms with higher $R_i\alpha_i$ values, makes investing viable for the *weaker* firms, i.e., ones with smaller $R_i\alpha_i$ values. Hence, tightening the budget constraints increases the number of active firms.

4.1.2 Effect on Equilibrium Investment

The analysis of the market activity in the equilibrium shows that the partition $(J(B), K(B), M(B))$ is piecewise constant in B_i. On an interval of B_i where $(J(B), K(B), M(B)) = (J_t, K_t, M_t)$ is constant, $F(B) = F_t(\rho, B)$ (by Theorem 3.2) is given by (23), which assures that $F(B)$ is continuous and increasing in B_i on the given interval (since θ_t is increasing in B_i). The continuity at the endpoints follows from Lemma 3.1. On any interval such that $(J(B), K(B), M(B))$ is constant, the derivative of $F(B)$ with respect to B_i is

$$\frac{\partial F(B)}{\partial B_i} = \begin{cases} \frac{\alpha_i}{\sqrt{v_t^2+4\gamma_t(\theta_t+\zeta_t+\rho)}}, & \text{if } i \in J(B), \\ 0, & i \in N \setminus J(B). \end{cases} \tag{28}$$

Since (28) is nonnegative, $F(B)$ is increasing in B_i for all $i \in N$. Clearly, a reduction in the budget of a firm affects the equilibrium if and only if the reduced budget of the firm is below its equilibrium investment before the reduction, i.e., if and only if $i \in J(B)$. If $i \in J(B) = J_t$ and $v_t \neq 0$, then

$$\frac{\partial F(B)}{\partial B_i} \leq \frac{\alpha_i}{|v_t|} \leq \alpha_i, \tag{29}$$

where $|v_t|$ represents the absolute value of v_t.

Let $\bar{x}_i^*$ be the equilibrium investment of firm i when B_j's are fixed for $j \neq i$ and $B_i = \infty$. From the discussion of the market activity and Eq. (26), it follows that $x_i^*(B)$ is strictly increasing in B_i on the interval $[L_i, \bar{x}_i^*]$ and constant thereon (equals $\bar{x}_i^*$).

To analyze the variation of $x_j^*(B)$ as a function of B_i for $i \neq j$, express $x_j^*(B)$ as in Corollary 3.1:

$$x_j^*(B) = \begin{cases} L_j, & \text{if } \frac{F(B)(R_j\alpha_j - F(B))}{R_j\alpha_j^2} \leq L_j, \\ \frac{F(B)(R_j\alpha_j - F(B))}{R_j\alpha_j^2}, & \text{if } L_j < \frac{F(B)(R_j\alpha_j - F(B))}{R_j\alpha_j^2} < B_j, \\ B_j, & \text{if } \frac{F(B)(R_j\alpha_j - F(B))}{R_j\alpha_j^2} \geq B_j. \end{cases} \tag{30}$$

Since $F(B)$ is increasing in B_i, $\frac{F(B)(R_j\alpha_j - F(B))}{R_j\alpha_j^2}$ is increasing in B_i for $F(B) \leq R_j\alpha_j/2$ and decreasing in B_i for $F(B) > R_j\alpha_j/2$. The direction of change is preserved by (30), so the equilibrium investment of firm j is first increasing then decreasing in the budget of firm $i \neq j$. The following example illustrates that unlike the equilibrium investment of firm i, the equilibrium investment of firm j is not necessarily monotonic in the budget of firm $i \neq j$.

Example 4.1 Consider two firms such that $\alpha_1 = 0.1$, $R_1 = 100$, $R_2 = \alpha_2 = 1$, and $L_1 = L_2 = 0$. Let the second firm have an infinite budget and vary the budget of the first one. At the interest rate $\rho = 0.1$, $x_2^*(B) = 0$ for $B \geq (9, \infty)$, $x_2^*(8, \infty) = 0.049$, $x_2^*(2, \infty) = 0.248$ and $x_2^*(0.5, \infty) = 0.237$. The investment of firm 2 first increases then decreases as the budget of the first firm is reduced.

4.1.3 *Effect on Equilibrium Utility*

In a market with a single firm, introducing any constraints reduces the maximum utility that the firm can achieve. In particular, the utility decreases as the firm's budget decreases. The existence of multiple firms (and therefore the presence of competition) in the market complicates this analysis.

• *Effect of B_i on equilibrium utility of $j \neq i$.*

Case 1: $j \in J(B) \cup K(B)$. By (27) and since $F(B)$ is increasing in B_i, $U_j^*(B)$ decreases with B_i on any interval of B_i where $(J(B), K(B), M(B))$ is constant, for $j \in J(B) \cup K(B)$.

Case 2: $j \in M(B)$. In this case,

$$\frac{\partial U_j^*(B)}{\partial B_i} = \left(\frac{-2}{\alpha_j}\right)\left(1 - \frac{F(B)}{R_j \alpha_j}\right)\left(\frac{\partial F(B)}{\partial B_i}\right).$$

Since $j \in M(B)$ and $L_j \geq 0$, (30) implies that $F(B) < R_j \alpha_j$. If also $i \in J(B)$, then $\frac{\partial F(B)}{\partial B_i} > 0$ so that $U_j^*(B)$ is decreasing in B_i; otherwise, $U_j^*(B)$ remains the same.

Hence, a firm never suffers from a reduction in the budget of another firm.

• *Effect of B_i on equilibrium utility of i.*

Case 1: $i \in N \setminus J(B)$. By (27) and (30), if $i \in N \setminus J(B)$, then a reduction in its budget does not affect its equilibrium utility.

Case 2: $i \in J(B) = J_t$ *and* $v_t \neq 0$.

$$\frac{\partial U_i^*(B)}{\partial B_i} \geq \frac{R_i \alpha_i}{F(B)} - 1 - \frac{B_i R_i \alpha_i^2}{F^2(B)} = -\frac{F^2(B) - R_i \alpha_i F(B) - R_i \alpha_i^2 B_i}{F^2(B)} \geq 0,$$

where the first inequality follows from (29) and the second inequality from (30).

Case 3: $i \in J(B) = J_t$ *and* $v_t = 0$. The assumption $v_t = 0$ implies that M_t is a singleton, equivalently, all firms but one (say firm m) invest either their lower bounds or their budgets. Since $i \in J_t$, it follows that $m \neq i$. By (22), $\gamma_t = \frac{1}{R_m \alpha_m}$ and by (23), $F(B) = \sqrt{R_m \alpha_m (\theta_t + \zeta_t + \rho)}$ and

$$\frac{\partial U_i^*(B)}{\partial B_i} = \frac{R_i \alpha_i}{F(B)} - 1 - \frac{B_i R_i \alpha_i^2 R_m \alpha_m}{2F^3(B)} \tag{31}$$

$$= \frac{R_i \alpha_i R_m \alpha_m}{2F^3(B)}\left[\left(\frac{2F(B)}{R_m \alpha_m}\right)\frac{F(B)(R_i \alpha_i - F(B))}{R_i \alpha_i} - B_i \alpha_i\right] \tag{32}$$

$$= \frac{R_m \alpha_m}{2F^3(B)}\left[2(\theta_t + \rho) R_i \alpha_i - 2(\theta_t + \rho) F(B) - B_i R_i \alpha_i^2\right]. \tag{33}$$

If $\frac{\partial U_i^*(B)}{\partial B_i} < 0$, then (33) implies $R_i\alpha_i < 2F(B)$ and (32) together with $i \in J(B)$ implies

$$\left(\frac{2F(B)}{R_m\alpha_m}\right)\frac{F(B)(R_i\alpha_i - F(B))}{R_i\alpha_i} < B_i\alpha_i \le \frac{F(B)(R_i\alpha_i - F(B))}{R_i\alpha_i},$$

so $R_m\alpha_m > 2F(B)$. Hence, for the equilibrium utility of firm i to be decreasing in its budget when $v_t = 0$, it is necessary to have $R_m\alpha_m > 2F(B) > R_i\alpha_i$. This can never happen when the firm which does not spend a boundary amount is weaker than firm i. However, the following example illustrates that this counterintuitive phenomenon, i.e., a decrease in the utility as a result of an increase in the budget, can happen when firm m is stronger than firm i.

Example 4.2 Consider two firms such that $\alpha_1 = \alpha_2 = 1$, $R_1 = 50$, $R_2 = 12$, and $L_1 = L_2 = 0$. Let $B_1 = \infty$ and vary B_2. At the interest rate $\rho = 1$, $F(\infty, 1) = 10$, $x_2^*(\infty, 1) = 1$ and $U_2^*(\infty, 1) = 0.2$ whereas $F(\infty, 1.2) = 10.488$, $x_2^*(\infty, 1.2) = 1.2$ and $U_2^*(\infty, 1.2) = 0.173$. Hence, an increase in the budget of firm 2 reduced the equilibrium utility of the weaker firm 2.

4.2 Marginal Effect of Lower Bounds

The effect of increasing the lower bounds can be explored following the same steps, after parameterizing all variables in $L := (L_1, \ldots, L_n)$. If L_i increases, then $\underline{T}_i$ increases and $\overline{T}_i$ decreases, so the interval of interest rates for which the lower bound of firm i is binding expands. Further, (28) holds with B replaced by L everywhere and $J(L)$ replaced by $K(L)$, so $F(L)$ is increasing in L_i. Results analogous to those for upper bounds can be derived for lower bounds using these observations.

5 Global Optimality

This section analyzes the effects of centralization on the model developed in Sect. 2. In the centralized variant of the model, a single decision maker decides on the investments of all firms with the objective of maximizing the sum of their utilities subject to the individual lower and upper bound constraints of the firms. The aggregate utility of all firms under a joint investment vector $x \ge 0$ is referred to as the *global utility* of x and equals

$$U(x) = \sum_{i=1}^{n} U_i(x) = \sum_{i=1}^{n} x_i\left(\frac{R_i\alpha_i}{\alpha^T x + \rho} - 1\right). \tag{34}$$

Let $L := (L_1, \ldots, L_n)$ and $B := (B_1, \ldots, B_n)$ be the nonnegative vectors of lower and upper bounds, respectively. A vector $L \le x^\star \le B$ is *globally optimal* if it maximizes $U(x)$ over all $L \le x \le B$, i.e.,

$$x^\star \in \operatorname*{arg\,max}_{L \le x \le B} U(x).$$

As discussed in Sect. 3, one can restrict attention to the case with $0 \le B_i \le \max\{L_i, R_i\}$ for every $i \in N$ (which implies that all B_i's are finite), since otherwise, either the problem is infeasible or the upper bound constraint is redundant. For the relaxed model where $L_i = 0$ and $B_i = \infty$ (or equivalently, $B_i = R_i$) for all $i \in N$, [1] showed that there exists a globally optimal solution in which at most one firm invests a positive amount. The analysis in the sequel extends this structure to the problem with lower and upper bounds on the firms' investments and develops an efficient method to compute a globally optimal solution.

Lemma 5.1

(i) *There exists a globally optimal solution in which at most one firm invests strictly between its lower and upper bounds.*
(ii) *If $x^\star$ is a globally optimal solution and $x_i^\star > L_i$, then $R_i\alpha_i > \alpha^T x^\star + \rho(>\rho)$.*
(iii) *If $\rho \ge R_i\alpha_i$ for each $i \in N$, then L is a unique globally optimal solution. Further, if $L = 0$ is globally optimal, then $\rho \ge R_i\alpha_i$ for all $i \in N$.*

The forthcoming results of this section rely on the idea of dividing the problem into two subproblems: first to maximize the global utility for each fixed amount of total investment by optimally allocating it to firms, and second to maximize the global utility with respect to the total investment. This approach motivates the following definitions.

For $t = 0, \dots, n$, let $\omega_t := \sum_{k=1}^{t} \alpha_k B_k + \sum_{k=t+1}^{n} \alpha_k L_k$, where $\sum_{k=i}^{j} a_k = 0$ for $i > j$. Also for $t = 1, \dots, n$, define $\mathcal{I}_t := [\omega_{t-1}, \omega_t]$ and $\widehat{U}^t : [\omega_{t-1}, \infty[\to \mathbb{R}$ by

$$\widehat{U}^t(v) := \frac{\sum_{k=1}^{t-1} R_k\alpha_k B_k + \sum_{k=t}^{n} R_k\alpha_k L_k}{v+\rho} - \sum_{k=1}^{t-1} B_k - \sum_{k=t}^{n} L_k + (v - \omega_{t-1})\left(\frac{R_t}{v+\rho} - \frac{1}{\alpha_t}\right). \tag{35}$$

Also, let $\widehat{U} : [\omega_0, \omega_n] \to \mathbb{R}$ be defined for $v \in [\omega_0, \omega_n]$ by

$$\widehat{U}(v) := \widehat{U}^t(v) \quad \text{for } v \in \mathcal{I}_t. \tag{36}$$

(Lemma 5.2(ii) below verifies that $\widehat{U}$ is well-defined at the ω_t's). Finally, let $U^\# : [\omega_0, \omega_n] \to \mathbb{R}$ be defined for $v \in [\omega_0, \omega_n]$ by

$$U^\#(v) := \max_{L \le x \le B,\ \alpha^T x = v} U(x). \tag{37}$$

Evidently, as U is continuous on the compact set $\{x \in \mathbb{R}^n : L \le x \le B, \alpha^T x = v\}$, standard arguments show that $U^\#$ is well defined and continuous.

A real-valued function is *unimodal* on an interval, if there exists no $x < y < z$ in its domain satisfying $f(x) \ge f(y)$ and $f(y) \le f(z)$, i.e., it does not allow a (weak) increase after a (weak) decrease. Equivalently, $f(x)$ is unimodal on an interval $[a, b]$ if there exists a scalar $y \in [a, b]$ such that $f(x)$ is strictly increasing on $[a, y]$ and strictly decreasing on $[y, b]$. Strictly concave, strictly decreasing, and strictly increasing functions are unimodal. Evidently, a continuous unimodal function has a unique maximizer on each bounded interval.

Henceforth in this section, whenever necessary, strict inequalities are imposed on data to preclude degenerate situations. Before addressing the general case, consider the following instances with a simple structure.

Lemma 5.2 *Suppose*

$$\frac{R_1}{v+\rho} - \frac{1}{\alpha_1} > \cdots > \frac{R_n}{v+\rho} - \frac{1}{\alpha_n} \quad \textit{for all } \omega_0 \le v \le \omega_n. \tag{38}$$

Then:

(i) *For $t \in N$, $\widehat{U}^t$ is strictly decreasing or strictly concave*; *in particular, it is unimodal.*
(ii) *For $t \in N \setminus \{n\}$ and $v \ge \omega_t$, $\widehat{U}^t(v) \ge \widehat{U}^{t+1}(v)$ with equality holding only at $v = \omega_t$, and*

$$\frac{d\widehat{U}^t(\omega_t)}{dv} > \frac{d\widehat{U}^{t+1}(\omega_t)}{dv}. \tag{39}$$

(iii) *For $v \in \mathcal{I}_t$, a unique solution of the maximization problem defining $U^\#(v)$* (*by* (37)) *is*

$$\left(B_1, \ldots, B_{t-1}, L_t + \frac{v - \omega_{t-1}}{\alpha_t}, L_{t+1}, \ldots, L_n\right)$$

and $U^\#(v) = \widehat{U}^t(v) = \widehat{U}(v)$, so, $U^\# = \widehat{U}$ on $[\omega_0, \omega_n]$.
(iv) *$U^\# = \widehat{U}$ is unimodal on $[\omega_0, \omega_n]$.*
(v) *If $v^\star \in \mathcal{I}_{t^\star}$ is the* (*unique*) *maximizer of $U^\#$ over $[\omega_0, \omega_n]$, then*

$$x^\star := \left(B_1, \ldots, B_{t^\star-1}, L_{t^\star} + \frac{v^\star - \omega_{t^\star-1}}{\alpha_{t^\star}}, L_{t^\star+1}, \ldots, L_n\right) \tag{40}$$

is a unique globally optimal solution.

Theorem 5.1 *Suppose* (38) *holds and let $t^\star$ be the first index $t \in N$ for which the global maximizer of $\widehat{U}^t$ on the interval $[\omega_{t-1}, \omega_n]$ is in $\mathcal{I}_t$. Then $t^\star \le |\{i \in N : R_i\alpha_i > \rho\}| + 1$, the corresponding maximizer $v^\star$ is the unique maximizer of $\widehat{U}$ over the interval $[\omega_0, \omega_n]$, and the unique globally optimal solution is $x^\star := (B_1, \ldots, B_{t^\star-1}, L_{t^\star} + \frac{v^\star - \omega_{t^\star-1}}{\alpha_{t^\star}}, L_{t^\star+1}, \ldots, L_n)$.*

When (38) holds, Theorem 5.1 yields an efficient method to compute the unique globally optimal solution. Starting with $t = 1$, find the unique maximizer v^t of $\widehat{U}^t$ over $[\omega_{t-1}, \infty[$. If $v^t \le \omega_t$, then use Theorem 5.1 to construct the unique globally optimal solution and stop; else, replace t by $t + 1$ and iterate. Each iteration requires finitely many operations, which include the taking of a square root and Theorem 5.1 states that the number of iterations is bounded by $|\{i \in N : R_i\alpha_i > \rho\}| + 1$, and in the worst case by n. The number of iterations can be reduced further by employing bisection. The function $\widehat{U}$ was shown to be unimodal and after finding its maximum $v^\star$, a globally optimal solution is easily constructed using Theorem 5.1. There are n intervals. Let $a=1$, $b=n$, and $t=\lfloor\frac{a+b}{2}\rfloor$. Find the unique maximizer v^t of the extension of $\widehat{U}^t$ to the real line. If $v^t\geq\omega_t$, then $\widehat{U}$ is strictly increasing on the left side of ω_t, so $v^\star\geq\omega_t$. In this case, let $a=t$ and reiterate. Alternatively, if $v^t<\omega_t$, then $\widehat{U}$ is strictly decreasing at ω_t, so $v^\star<\omega_t$ and in this case, let $b=t$ and repeat. The total effort required to find a globally optimal solution is then $O(\log n)$.

The next lemma applies to the general model (without (38)) and its corollary provides sufficient conditions for (38).

Lemma 5.3 *Suppose*

$$\frac{R_i}{\rho}-\frac{1}{\alpha_i}>\frac{R_j}{\rho}-\frac{1}{\alpha_j}. \tag{41}$$

(i) *If either* $[R_i\leq R_j]$ *or* $[R_i>R_j$ *and* $\alpha_i\geq\alpha_j]$, *then*

$$\frac{R_i}{v+\rho}-\frac{1}{\alpha_i}>\frac{R_j}{v+\rho}-\frac{1}{\alpha_j}\quad\textit{for all } v\geq 0. \tag{42}$$

(ii) *If* $[R_i>R_j$ *and* $\alpha_i<\alpha_j]$, *then for* $v_{ij}:=\frac{R_i-R_j}{\frac{1}{\alpha_i}-\frac{1}{\alpha_j}}-\rho>0$,

$$\left[\frac{R_i}{v+\rho}-\frac{1}{\alpha_i}>\frac{R_j}{v+\rho}-\frac{1}{\alpha_j}\right]\quad\Leftrightarrow\quad[0\leq v<v_{ij}], \tag{43}$$

$$\left[\frac{R_i}{v+\rho}-\frac{1}{\alpha_i}<\frac{R_j}{v+\rho}-\frac{1}{\alpha_j}\right]\quad\Leftrightarrow\quad[v>v_{ij}], \tag{44}$$

$$\left[\frac{R_i}{v_{ij}+\rho}-\frac{1}{\alpha_i}=\frac{R_j}{v_{ij}+\rho}-\frac{1}{\alpha_j}\right]\quad\Leftrightarrow\quad[v=v_{ij}]. \tag{45}$$

Corollary 5.1 *If either*

$$R_1\geq\cdots\geq R_n,\qquad \alpha_1\geq\cdots\geq\alpha_n\quad\textit{and}\quad R_1\alpha_1>\cdots>R_n\alpha_n, \tag{46}$$

or

$$\alpha_1\leq\cdots\leq\alpha_n\quad\textit{and}\quad R_1\alpha_1>\cdots>R_n\alpha_n, \tag{47}$$

or

$$v_{ij}>\sum_{k=1}^{n}\alpha_k B_k\quad\textit{for all } i,j\in N \textit{ such that } \frac{R_i}{\rho}-\frac{1}{\alpha_i}>\frac{R_j}{\rho}-\frac{1}{\alpha_j}, \tag{48}$$

then (38) *and the conclusions of Theorem* 5.1 *hold.*

If $R_i\alpha_i\leq\rho$, then firm i will not be active in the globally optimal solution and it can be ignored. Hence, the conditions in Corollary 5.1 need to be checked only for the subset of firms with $R_i\alpha_i>\rho$.

For the relaxed problem (where all the firms have an infinite budget or equivalently $B_i>R_i/4$ for all i), [1] proved that if the globally optimal solution is not zero, then all the investment is allocated to the firm with the highest value of $\sqrt{R_i} - \sqrt{\frac{\rho}{\alpha_i}}$ (assuming the maximizer is unique). If either (46) or (47) holds, then the ranking with respect to this criterion coincides with the ranking with respect to the value $\frac{R_i}{\rho} - \frac{1}{\alpha_i}$. However, if these assumptions are not satisfied, the two rankings are not necessarily the same as the following example illustrates.

Example 5.1 Let $R_1 = 81$, $R_2 = 100$, $\alpha_1 = 4$, $\alpha_2 = 1$, and $\rho = 5$. Then $\frac{R_1}{\rho} - \frac{1}{\alpha_1} < \frac{R_2}{\rho} - \frac{1}{\alpha_2}$ and $\sqrt{R_1} - \frac{\sqrt{\rho}}{\sqrt{\alpha_1}} > \sqrt{R_2} - \frac{\sqrt{\rho}}{\sqrt{\alpha_2}}$.

It is worth noting that a discrepancy between the two aforementioned rankings of firms i and j is possible and relevant if and only if $R_i\alpha_i > R_j\alpha_j > \rho$, $\alpha_i > \alpha_j$, and $R_j > R_i$. Example 5.1 suggests that allocating the investment according to the index $\frac{R_i}{\rho} - \frac{1}{\alpha_i}$ is not always optimal for the general case. The index $\sqrt{R_i} - \sqrt{\frac{\rho}{\alpha_i}}$ works for the unconstrained problem, so one might think that it would work in the constrained case, yet the following example shows that it does not.

Example 5.2 Consider two firms with $L_1 = L_2 = 0$, $R_1 = 4$, $\alpha_1 = R_2 = \rho = 1$, and $\alpha_2 = 25$. From the results of Canbolat et al. (2012), if $B_1 = B_2 = \infty$, the unique globally optimal solution is $x^\star = (1, 0)$ since $\sqrt{R_1} - \frac{\sqrt{\rho}}{\sqrt{\alpha_1}} = 1 > \sqrt{R_2} - \frac{\sqrt{\rho}}{\sqrt{\alpha_2}} = 0.8$. If the upper bound of the first firm is reduced to 0.25, Lemma 5.1(i) assures that there exists a globally optimal solution in which either exactly one of the firms invests a positive amount or firm 1 invests its full budget. The maximum utility when firm 1 invests its full budget is $U(0.25, 0.05) = 0.6$, the maximum utility when only firm 1 invests, is $U(0.25, 0) = 0.55$ and the maximum utility when only firm 2 invests is $U(0, 0.16) = 0.64$. Thus, $(0, 0.16)$ is a globally optimal solution; in particular, it is the unique globally optimal solution. In this case, the introduction of a budget constraint replaced the first firm by the second firm as the unique globally optimal monopoly. This demonstrates the nonmonotonicity of the globally optimal solution as a function of the budgets. Letting $x^\star(B)$ a globally optimal solution for the budget vector B, $x^\star(0.25, \infty) = (0, 0.16) \nleq (1, 0) = x^\star(\infty, \infty)$.

Lemma 5.3 asserts that, generally, the ranking of the firms with respect to the index $\frac{R_i}{v+\rho} - \frac{1}{\alpha_i}$ depends on the value $v \geq 0$. The rest of this section records extensions of Lemma 5.2 and Theorem 5.1 to situations where (38) holds only for v in a subinterval of $[\omega_0, \omega_n]$.

Lemma 5.1′ *Suppose $\omega_0 \leq a < b \leq \omega_n$ and (38) holds for $v \in [a, b]$. Then:*

(i) *For $t \in N$, $\widehat{U}^t$ is strictly decreasing or strictly concave on $[\omega_{t-1}, \infty] \cap [a, b]$; in particular, it is unimodal on $[\omega_{t-1}, \infty] \cap [a, b]$.*
(ii) *For $t \in N \setminus \{n\}$ and $v \in [\omega_t, \infty] \cap [a, b]$, $\widehat{U}^t(v) \geq \widehat{U}^{t+1}(v)$ with equality holding only at $v = \omega_t$, and if $\omega_t \in [a, b]$, then (39) holds.*

(iii) *For* $v \in \mathcal{I}_t \cap [a,b]$, *a unique solution of the maximization problem defining* $U^{\#}(v)$ (*by* (37)) *is*

$$\left(B_1, \dots, B_{t-1}, L_t + \frac{v - \omega_{t-1}}{\alpha_t}, L_{t+1}, \dots, L_n\right)$$

and $U^{\#}(v) = \widehat{U}^t(v) = \widehat{U}(v)$; *in particular,* $U^{\#} = \widehat{U}$ *on* $[a,b]$.

(iv) $U^{\#} = \widehat{U}$ *is unimodal on* $[a,b]$.

(v) *If* $v^\star \in \mathcal{I}_{t^\star}$ *is a maximizer of* $U^{\#}$ *over* $[a,b]$, *then*

$$x^\star := \left(B_1, \dots, B_{t^\star-1}, L_{t^\star} + \frac{v^\star - \omega_{t^\star-1}}{\alpha_{t^\star}}, L_{t^\star+1}, \dots, L_n\right) \in \operatorname*{arg\,max}_{L \le x \le B,\ a \le \alpha^T x \le b} U(x);$$

further, if $v^\star$ *is a unique maximizer, then* $x^\star$ *is a unique maximizer, respectively.*

Theorem 5.1′ *Suppose* $\omega_0 \le a < b \le \omega_n$, (38) *holds for* $v \in [a,b]$ *and* $t^\star$ *is the first index* $t \in N$ *for which* $[\omega_{t-1}, \omega_n] \cap [a,b] \ne \emptyset$ *and the global maximizer of* $\widehat{U}^t$ *over this intersection is in* $\mathcal{I}_t$. *Then necessarily* $t^\star \le |\{i \in N : R_i \alpha_i > \rho\}| + 1$, *the corresponding maximizer* $v^\star$ *is the unique maximizer of* $\widehat{U}$ *over* $[a,b]$ *and the unique maximizer of* U *over* $\{x \in \mathbb{R}^n : L \le x \le B, a \le \alpha^T x \le b\}$ *is*

$$x^\star \equiv \left(B_1, \dots, B_{t^\star-1}, L_{t^\star} + \frac{v^\star - \omega_{t^\star-1}}{\alpha_{t^\star}}, L_{t^\star+1}, \dots, L_n\right).$$

Theorem 5.1′ suggests that when (38) holds for $v \in [a,b]$, the method discussed after Theorem 5.1 can be employed to find a unique maximizer of U over $\{x \in \mathbb{R}^n : L \le x \le B, a \le \alpha^T x \le b\}$. In fact, by renumbering indices, the method extends to situations the firms' indices satisfy for $v \in [a,b]$ a (fixed) coordinate-permutation of (38)—just renumber the indices. Thus,

$$\max_{L \le x \le B,\ a \le \alpha^T x \le b} U(x)$$

can be solved for every subinterval $[a,b]$ of $[\omega_0, \omega_n]$ in which the order of $\frac{R_1}{v+\rho} - \frac{1}{\alpha_1}, \dots, \frac{R_n}{v+\rho} - \frac{1}{\alpha_n}$ is invariant of $v \in [a,b]$. The maximization in each such subinterval involves $O(\log n)$ iterations.

For $v \le v_{ij}$, write $i \preceq^v j$ and call i *weakly below* j *under the* v*-ranking*. Since $v_{ij} = v_{ji}$ for $i \ne j$, there are at most $\binom{n}{2}$ distinct v_{ij}'s. Then let $\psi_0 := \omega_0$, $\Psi := \{\omega_0 < v_{ij} < \omega_n : i \preceq^0 j\}$, $\psi_{|\Psi|+1} := \omega_n$ and order the elements of Ψ in increasing order, i.e., $\psi_1 < \cdots < \psi_{|\Psi|}$. The $\preceq^v$-ranking of the firms is invariant of the value v for $v \in (\psi_{q-1}, \psi_q)$ and it extends to the endpoints of the intervals (which admit an additional ranking as well). The problem of finding a globally optimal solution can be decomposed into $|\Psi| + 1$ maximization problems as follows:

$$\max_{L \le x \le B} U(x) = \max_{t=0,\dots,|\Psi|} \left[\max_{L \le x \le B,\ \psi_t \le \alpha^T x \le \psi_{t+1}} U(x)\right].$$

The maximum utility and a (unique) maximizer can be found for each bracketed maximization problem with the method described in the previous paragraph. The maximum of these over all intervals is the globally optimal utility. Hence, a globally optimal solution can be found by solving $\binom{|\Psi|+1\leq n}{2+1}$ problems, each of which requires an effort of $O(\log n)$, so the total computational effort is $O(n^2 \log n)$.

6 Extensions

The results of this paper generalize to the following cases.

Generalized Utility Suppose the utility of firm i as a function of the vector $\tilde{x} \in \mathbb{R}^n$ is instead

$$\tilde{U}_i(\tilde{x}) = \frac{\beta_i \tilde{x}_i + K_i}{\alpha^T \tilde{x} + \tilde{\rho}} - \tilde{x}_i. \tag{49}$$

Maximizing this utility with respect to $0 \leq \tilde{L}_i \leq \tilde{x}_i \leq \tilde{B}_i$ is equivalent to maximizing

$$U_i(x) = \frac{\beta_i x_i}{\alpha^T x + \rho} - x_i \tag{50}$$

over $L_i \leq x_i \leq B_i$, where $\rho := \tilde{\rho} - \sum_{i\in N} \frac{\alpha_i K_i}{\beta_i}$, $L_i := \tilde{L}_i + \frac{K_i}{\beta_i}$ and $B_i := \tilde{B}_i + \frac{K_i}{\beta_i}$. For $x \in \mathbb{R}^n$ such that $x_i \geq L_i$, $\alpha^T x + \rho \geq \sum_{i\in N} \alpha_i L_i + \tilde{\rho} - \sum_{i\in N} \frac{\alpha_i K_i}{\beta_i} \geq \tilde{\rho} > 0$, so even though ρ can be negative, the concavity of $U_i(x)$ in x_i is preserved on the feasible region.

Zero Interest Rate The results obtained for the discounted model also hold for the undiscounted model with two or more firms. In this case, the utility functions exhibit discontinuity at the origin, but once the null vector is eliminated from the set of candidates for Nash equilibria, the argument for positive interest rates holds. The detailed extension is done in [1] for the relaxed model.

Spreading Payments The capital investment that is assumed to be made upfront can be replaced by a fixed investment cost spread over a finite time horizon, say T_i (with commitment to continue to pay even if the competition is lost). Then the discounted investment cost becomes $\int_0^{T_i} (\frac{x_i}{T_i}) e^{-\rho t}\, dt = \beta_i x_i$ for a positive constant β_i; the second term of the utility of function of firm i will then be $-\beta_i x_i$ rather than $-x_i$. Changing the decision variable to $x_i' = \beta_i x_i$, letting $\alpha_i' := \frac{\alpha_i}{\beta_i}$ and $B_i' := \beta_i B_i$ reduces the problem to one studied in this paper. Our results will apply with α_i''s replacing α_i's and B_i''s replacing the B_i's if an upper bound is imposed on the total investment. However, if there are temporal budget constraints, i.e., individual upper bounds for each time period, then the problem becomes more complicated and requires dynamic analysis.

Renewable Resources Alternative investment models involve "renewable resources" rather than upfront payments. In such models, a firm commits to make payments at a fixed rate throughout the lifetime of the project (i.e., until exactly one of the firms completes the project); see [1]. In particular, the utility of firm i with a predetermined investment rate of x_i (and with its completion time exponential with rate $\alpha_i x_i$)

is $\frac{R_i\alpha_i x_i - x_i}{\sum_{k=1}^n \alpha_k x_k+\rho}$ and a Nash equilibrium need not exist even for the unconstrained problem—firm i will try to invest as much as possible if $R_i\alpha_i > 1$ (and none otherwise). The results in [4] and [5] apply to this variant.

Nonlinear Rates of Exponential Distributions The existence of Nash equilibria generalizes to the model where each firm's rate parameter is concave in its investment from the classical result in [3]. However, an explicit representation for that case is unlikely under general assumptions.

7 Conclusions

This paper studied a competitive decision model where several firms compete over the development of a certain project. This situation is commonly observed in markets where firms race in bringing out novelties. The analysis in this paper complements the work in [1] by incorporating lower and upper bound constraints. The scarcity of resources, which is of interest particularly when the firms compete over multiple simultaneous R&D projects (see [2]), is accounted for through upper bounds while lower bounds may represent governmental constraints or early commitments.

Acknowledgements This research was partially supported by the Daniel Rose Yale University-Technion Initiative for Research on Homeland Security and Counter-Terrorism.

Appendix: Proofs

Proof of Theorem 3.1 If $M \neq \emptyset$, then (9) is a quadratic equation in which z^0 has a negative coefficient and z^2 has a positive coefficient; hence, it has a unique positive root. Alternatively, if $M = \emptyset$, then (9) is the linear equation $z - (\sum_{i\in J} B_i\alpha_i + \sum_{i\in K} L_i\alpha_i + \rho) = 0$, which has a unique (positive) root.

Consider a quadruple $(F, J, K, M) \in \mathcal{Q}$ and x^* defined by (10). Then (10) and (9) imply that

$$\alpha^T x^* + \rho = \sum_{k\in J} B_k\alpha_k + \sum_{k\in K} L_k\alpha_k + |M|F - F^2 \sum_{k\in M} \frac{1}{R_k\alpha_k} + \rho = F. \qquad (51)$$

To prove that x^* is a Nash equilibrium, it is enough to verify (4)–(8) for corresponding scalars $\tau_1, \dots, \tau_n, \sigma_1, \dots, \sigma_n$. First observe that (10) and (b) assure $L_i \le x_i^* \le B_i$ for $i \in N$, verifying (5). For $i \in J$, let $\tau_i = 0$ and $\sigma_i = \frac{R_i\alpha_i(F-\alpha_i B_i)}{F^2} - 1$. Then (7)–(8) are trivial and by (b), $R_i\alpha_i(F - \alpha_i B_i) \ge F^2$, implying that $\sigma_i \ge 0$; so, (6) holds. Next, by (51),

$$\frac{R_i\alpha_i(\alpha^T x^* - \alpha_i x_i^* + \rho)}{(\alpha^T x^* + \rho)^2} = \frac{R_i\alpha_i(F - \alpha_i B_i)}{F^2} = \sigma_i + 1,$$

which implies (4). For $i \in K$, let $\sigma_i = 0$ and $\tau_i = 1 - \frac{R_i\alpha_i(F-\alpha_i L_i)}{F^2}$. Then (7)–(8) are trivial and by (b), $R_i\alpha_i(F - \alpha_i L_i) \le F^2$, implying that $\tau_i \ge 0$; so, (6) holds. By (51),

$$\frac{R_i\alpha_i(\alpha^T x^* - \alpha_i x_i^* + \rho)}{(\alpha^T x^* + \rho)^2} = \frac{R_i\alpha_i(F - \alpha_i L_i)}{F^2} = 1 - \tau_i,$$

implying (4). For $i \in M$, let $\tau_i = \sigma_i = 0$. Then (6)–(8) are trivial. Also, by (10), $F - \alpha_i x_i^* = \frac{F^2}{R_i\alpha_i}$ and by (51), $\frac{R_i\alpha_i(\alpha^T x^* - \alpha_i x_i^* + \rho)}{(\alpha^T x^* + \rho)^2} = \frac{R_i\alpha_i(F-\alpha_i x_i^*)}{F^2} = 1$, verifying (4). Finally, showing that (F, J, K, M) satisfies (11)–(14) proves that the correspondence defined by (10) is one-to-one. Indeed, (51) verifies (11). Also, (10) and (b) imply that $x_i^* = B_i$ for all $i \in J$, $x_i^* = L_i$ for all $i \in K$ and $L_i < x_i^* < B_i$ for all $i \in M$. As J, K, M partition N (by (b)), $L_i \le x_i^* \le B_i$ for each $i \in N$, so (12)–(14) follow.

Next, consider a Nash equilibrium x^*. To show that (F, J, K, M) given by (11)–(14) belongs to $\mathcal{Q}$, use the fact that x^* must satisfy (4)–(8) with some $\tau_1, \dots, \tau_n, \sigma_1, \dots, \sigma_n$. For $i \in M$, (14) and (7)–(8) assure that $\tau_i = \sigma_i = 0$; hence, by (4) and (11),

$$F^2 = R_i\alpha_i\bigl(\alpha^T x^* - \alpha_i x_i^* + \rho\bigr) = R_i\alpha_i\bigl(F - \alpha_i x_i^*\bigr). \tag{52}$$

Since $x_i^* = B_i$ for $i \in J$ and $x_i^* = L_i$ for $i \in K$, $\sum_{i\in M}\alpha_i x_i^* = F - (\sum_{i\in J} B_i\alpha_i + \sum_{i\in K} L_i\alpha_i + \rho)$. Dividing (52) by $R_i\alpha_i$ and summing over $i \in M$ gives

$$\biggl(\sum_{i\in M}\frac{1}{R_i\alpha_i}\biggr)F^2 = |M|F - \sum_{i\in M}\alpha_i x_i^* = \bigl(|M|-1\bigr)F + \sum_{i\in J} B_i\alpha_i + \sum_{i\in K} L_i\alpha_i + \rho.$$

So F satisfies (9), verifying Condition (a). To prove Condition (b), first observe that for $i \in J$, (7) implies that $\tau_i = 0$; this and (4) imply that $\frac{R_i\alpha_i(F-\alpha_i x_i^*)}{F^2} = 1 + \sigma_i \ge 1$, so $B_i = x_i^* \le \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2}$. For $i \in K$, (8) implies that $\sigma_i = 0$; this together with (4) implies that $\frac{R_i\alpha_i(F-\alpha_i x_i^*)}{F^2} = 1 - \tau_i \le 1$, so $L_i = x_i^* \ge \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2}$. For $i \in M$, by (7) and (8), $\tau_i = \sigma_i = 0$; so $L_i < x_i^* = \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} < B_i$. As (12)–(14) and $L \le x \le B$ assure that J, K, M partition N, (b) follows and so, $(F, J, K, M) \in \mathcal{Q}$. Finally, to verify that the correspondence defined by (10) is onto and that (11)–(14) define its inverse, it is enough to show that x^* is the image of (F, J, K, M) under (10). Indeed, for $i \in M$, (52) implies that $x_i^* = \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2}$; as $x_i^* = B_i$ for $i \in J$ and $x_i^* = L_i$ for $i \in K$, x^* satisfies (10). □

Proof of Corollary 3.1 The corollary follows immediately from (10) and Condition (b). □

Proof of Corollary 3.2 (i) Suppose $x_i^* > L_i$, $R_j\alpha_j > R_i\alpha_i$ and $L_j = 0$. By (15),

$$\frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} > L_i \ge 0,$$

implying that $R_j\alpha_j > R_i\alpha_i > F$ and $\frac{F(R_j\alpha_j - F)}{R_j\alpha_j^2} > 0 = L_j$. Since also $B_j > 0$, it follows that $x_j^* > 0 = L_j$. (ii) Multiply (15) by α_i:

$$\alpha_i x_i^* = \begin{cases} \alpha_i L_i & \text{if } \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} \le L_i, \\ \frac{F(R_i\alpha_i - F)}{R_i\alpha_i} & \text{if } L_i < \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} < B_i, \\ \alpha_i B_i & \text{if } \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} \ge B_i. \end{cases}$$

Suppose $x_i^* > L_i$, $R_j\alpha_j > R_i\alpha_i$ and $B_j\alpha_j > B_i\alpha_i$. Then

$$\alpha_j x_j^* = \max\left\{\alpha_i L_i, \min\left\{F\left(1-\frac{F}{R_j\alpha_j}\right), \alpha_j B_j\right\}\right\}$$
$$\geq \min\left\{F\left(1-\frac{F}{R_j\alpha_j}\right), \alpha_j B_j\right\} > \min\left\{F\left(1-\frac{F}{R_i\alpha_i}\right), \alpha_i B_i\right\} = \alpha_i x_i^*.$$

(iii) If $x_i^* > L_i$, then by (15), $\frac{F(R_i\alpha_i-F)}{R_i\alpha_i^2} > L_i$. If $R_j\alpha_j > R_i\alpha_i$, $\alpha_j < \alpha_i$ and $L_j \leq L_i$ for $j \in N$, then $\frac{F(R_j\alpha_j-F)}{R_j\alpha_j^2} > \frac{F(R_i\alpha_i-F)}{R_i\alpha_i^2} > L_i \geq L_j$. Since also $B_j > L_j$, by (15), $x_j^* > L_j$. □

Proof of Lemma 3.1 (i) Let $1 \leq t \leq |\Phi|$. If $\upsilon_t = -1$, then $F_t(\rho) = \theta_t + \zeta_t + \rho$, so ρ_t is trivially in the extended domain of $F_t(\cdot)$. Alternatively, if $\upsilon_t \geq 0$, then

$$\upsilon_t^2 + 4\gamma_t(\theta_t + \zeta_t + \rho_t) = \upsilon_t^2 + 4\gamma_t\left(\gamma_t\varphi_t^2 - \upsilon_t\varphi_t\right) = (\upsilon_t - 2\gamma_t\varphi_t)^2 \geq 0,$$

assuring that ρ_t is in the extended domain of $F_t(\cdot)$.

Next, verify that ρ_t is in the extended domain of $F_{t-1}(\cdot)$. This is trivial if $\upsilon_{t-1} = -1$. Alternatively, if $\upsilon_{t-1} \geq 0$, then necessarily $t \geq 2$ and the following equality holds:

$$\theta_{t-1} + \zeta_{t-1} + \rho_t = \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t. \tag{53}$$

Recall that $\varphi_t \in \Phi = [\cup\{\{\overline{S}_i, \underline{S}_i\} : i \in N \text{ with } B_i < R_i/4\}] \cup [\cup\{\{\overline{T}_i, \underline{T}_i\} : i \in N\}]$, so the proof of (53) is divided in four cases.

Case 1: If $\varphi_t = \overline{T}_m$, then $J_t = J_{t-1}$, $K_t = K_{t-1} \setminus \{m\}$, $M_t = M_{t-1} \cup \{m\}$, $\upsilon_t = \upsilon_{t-1} + 1$, $\gamma_t = \gamma_{t-1} + \frac{1}{R_m\alpha_m}$, $\theta_t = \theta_{t-1}$, $\zeta_t = \zeta_{t-1} - L_m\alpha_m$, $\varphi_t^2 - R_m\alpha_m\varphi_t + R_m\alpha_m^2 L_m = 0$ (by (17)) and

$$\begin{aligned}\theta_{t-1} + \zeta_{t-1} + \rho_t &= \theta_t + (\zeta_t + L_m\alpha_m) \\ &\quad + \left[\left(\gamma_{t-1} + \frac{1}{R_m\alpha_m}\right)\varphi_t^2 - (\upsilon_{t-1}+1)\varphi_t - \theta_t - \zeta_t\right] \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t + \left(\frac{1}{R_m\alpha_m}\varphi_t^2 - \varphi_t + L_m\alpha_m\right) \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t,\end{aligned}$$

verifying (53).

Case 2: If $\varphi_t = \underline{T}_m$, then $J_t = J_{t-1}$, $K_t = K_{t-1} \cup \{m\}$, $M_t = M_{t-1} \setminus \{m\}$, $\upsilon_t = \upsilon_{t-1} - 1$, $\gamma_t = \gamma_{t-1} - \frac{1}{R_m\alpha_m}$, $\theta_t = \theta_{t-1}$, $\zeta_t = \zeta_{t-1} + L_m\alpha_m$, $\varphi_t^2 - R_m\alpha_m\varphi_t + R_m\alpha_m^2 L_m = 0$ (by (17)) and

$$\begin{aligned}\theta_{t-1} + \zeta_{t-1} + \rho_t &= \theta_t + (\zeta_t - L_m\alpha_m) \\ &\quad + \left[\left(\gamma_{t-1} - \frac{1}{R_m\alpha_m}\right)\varphi_t^2 - (\upsilon_{t-1}-1)\varphi_t - \theta_t - \zeta_t\right] \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t - \left(\frac{1}{R_m\alpha_m}\varphi_t^2 - \varphi_t + L_m\alpha_m\right) \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t,\end{aligned}$$

verifying (53).

Case 3: If $\varphi_t = \overline{S}_m$ and $B_m < R_m/4$, then $J_t = J_{t-1} \cup \{m\}$, $K_t = K_{t-1}$, $M_t = M_{t-1} \backslash \{m\}$, $\upsilon_t = \upsilon_{t-1} - 1$, $\gamma_t = \gamma_{t-1} - \frac{1}{R_m \alpha_m}$, $\theta_t = \theta_{t-1} + B_m \alpha_m$, $\zeta_t = \zeta_{t-1}$, $\varphi_t^2 - R_m \alpha_m \varphi_t + R_m \alpha_m^2 B_m = 0$ (by (16)) and

$$\begin{aligned}\theta_{t-1} + \zeta_{t-1} + \rho_t &= (\theta_t - B_m \alpha_m) + \zeta_t \\ &\quad + \left[\left(\gamma_{t-1} - \frac{1}{R_m \alpha_m} \right) \varphi_t^2 - (\upsilon_{t-1} - 1)\varphi_t - \theta_t - \zeta_t \right] \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t - \left(\frac{1}{R_m \alpha_m} \varphi_t^2 - \varphi_t + B_m \alpha_m \right) \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t,\end{aligned}$$

verifying (53).

Case 4: If $\varphi_t = \underline{S}_m$ and $B_m < R_m/4$, then $J_t = J_{t-1} \backslash \{m\}$, $K_t = K_{t-1}$, $M_t = M_{t-1} \cup \{m\}$, $\upsilon_t = \upsilon_{t-1} + 1$, $\gamma_t = \gamma_{t-1} + \frac{1}{R_m \alpha_m}$, $\theta_t = \theta_{t-1} - B_m \alpha_m$, $\zeta_t = \zeta_{t-1}$, $\varphi_t^2 - R_m \alpha_m \varphi_t + R_m \alpha_m^2 B_m = 0$ (by (16)) and

$$\begin{aligned}\theta_{t-1} + \zeta_{t-1} + \rho_t &= (\theta_t + B_m \alpha_m) + \zeta_t \\ &\quad + \left[\left(\gamma_{t-1} + \frac{1}{R_m \alpha_m} \right) \varphi_t^2 - (\upsilon_{t-1} + 1)\varphi_t - \theta_t - \zeta_t \right] \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t + \left(\frac{1}{R_m \alpha_m} \varphi_t^2 - \varphi_t + B_m \alpha_m \right) \\ &= \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t,\end{aligned}$$

completing the verification of (53) in all four cases. It now follows from (53) that

$$\begin{aligned}&\upsilon_{t-1}^2 + 4\gamma_{t-1}(\theta_{t-1} + \zeta_{t-1} + \rho_t) \\ &\quad = \upsilon_{t-1}^2 + 4\gamma_{t-1}\big(\gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t\big) = (\upsilon_{t-1} - 2\gamma_{t-1}\varphi_t)^2 \geq 0,\end{aligned}$$

proving that ρ_t is in the extended domain of $F_{t-1}(\cdot)$.

Next is the proof of the two equalities in (i). When $\upsilon_t \geq 0$, $F_t(\rho) = \varphi_t$ if and only if

$$\sqrt{\upsilon_t^2 + 4\gamma_t(\theta_t + \zeta_t + \rho)} = 2\gamma_t \varphi_t - \upsilon_t,$$

or equivalently (by squaring both sides, rearranging and dividing by $4\gamma_t$),

$$\rho = \gamma_t \varphi_t^2 - \upsilon_t \varphi_t - \theta_t - \zeta_t = \rho_t, \tag{54}$$

the last equality holding by (24); in particular, $F_t(\rho_t) = \varphi_t$. Similarly, when $\upsilon_{t-1} \geq 0$, $F_{t-1}(\rho) = \varphi_t$ if and only if

$$\rho = \gamma_{t-1}\varphi_t^2 - \upsilon_{t-1}\varphi_t - \theta_{t-1} - \zeta_{t-1} = \rho_t,$$

the last equality following from (53); in particular, $F_{t-1}(\rho_t) = \varphi_t$. Next, if $\upsilon_t = -1$, then $\gamma_t = 0$ and $\varphi_t = F_t(\rho) = \theta_t + \rho$ is trivially equivalent to (54), so $F_t(\rho_t) = \varphi_t$. Finally, assume that $\upsilon_{t-1} = -1$ (i.e., $M_{t-1} = \emptyset$), in which case, $\upsilon_t = 0$ and either

$\varphi_t = \underline{S}_m$ for some $m \in N$ with $B_m < R_m/4$ or $\varphi_t = \overline{T}_m$ for some $m \in N$. If $\varphi_t = \underline{S}_m$ for some $m \in N$ with $B_m < R_m/4$, then $\theta_t = \theta_{t-1} - B_m\alpha_m$, $\zeta_t = \zeta_{t-1}$, $\upsilon_t = 0$, $\gamma_t = \frac{1}{R_m\alpha_m}$, (by (16)) $\varphi_t^2 - R_m\alpha_m\varphi_t + R_m\alpha_m^2 B_m = 0$ and

$$F_{t-1}(\rho_t) = \theta_{t-1} + \zeta_{t-1} + \rho_t = (\theta_t + B_m\alpha_m) + \zeta_t + \left(\gamma_t\varphi_t^2 - \upsilon_t\varphi_t - \theta_t - \zeta_t\right)$$

$$= B_m\alpha_m + \frac{\varphi_t^2}{R_m\alpha_m} = \varphi_t.$$

Alternatively, if $\varphi_t = \overline{T}_m$, then $\theta_t = \theta_{t-1}$, $\zeta_t = \zeta_{t-1} - L_m\alpha_m$, $\upsilon_t = 0$, $\gamma_t = \frac{1}{R_m\alpha_m}$, (by (17))

$$\varphi_t^2 - R_m\alpha_m\varphi_t + R_m\alpha_m^2 L_m = 0$$

and

$$F_{t-1}(\rho_t) = \theta_{t-1} + \zeta_{t-1} + \rho_t = \theta_t + (\zeta_t + L_m\alpha_m) + \left(\gamma_t\varphi_t^2 - \upsilon_t\varphi_t - \theta_t - \zeta_t\right)$$

$$= L_m\alpha_m + \frac{\varphi_t^2}{R_m\alpha_m} = \varphi_t.$$

(ii) For $t = 1, \ldots, |\Phi| - 1$, part (i) implies that $F_t(\rho_t) = \varphi_t > \varphi_{t+1} = F_t(\rho_{t+1})$. Since $F_t(\cdot)$ is strictly increasing, $\rho_t > \rho_{t+1}$. □

Proof of Theorem 3.2 (i) The definition of $F_t(\rho)$ assures that $(F, J, K, M) = (F_t(\rho), J_t, K_t, M_t)$ satisfies Condition (a). To verify Condition (b), consider two cases: $t = 0$ and $t \geq 1$. If $t = 0$, then $\varphi_1 = \rho_1 \leq \rho$, $\{i \in N : \frac{F_t(\rho)(R_i\alpha_i - F_t(\rho))}{R_i\alpha_i^2} \leq L_i\} = \{i \in N : F_t(\rho) > \overline{T}_i\} = \{i \in N : \varphi_t > \overline{T}_i\} = N = K_t$ and so $J_t = M_t = \emptyset$. If $t \geq 1$, then the strict monotonicity of F_t, part (i) of Lemma 3.1 and $\rho_{t+1} \leq \rho < \rho_t$ imply that $\varphi_{t+1} = F_t(\rho_{t+1}) \leq F_t(\rho) < F_t(\rho_t) = \varphi_t$, assuring that

$$\left\{i \in N : \frac{F_t(\rho)(R_i\alpha_i - F_t(\rho))}{R_i\alpha_i^2} \geq B_i\right\} = \{i \in N : \underline{S}_i < \varphi_t \leq \overline{S}_i \text{ and } B_i < R_i/4\} = J_t,$$

$$\left\{i \in N : \frac{F_t(\rho)(R_i\alpha_i - F_t(\rho))}{R_i\alpha_i^2} \leq L_i\right\} = \left\{i \in N : F_t(\rho) \leq \underline{T} \text{ or } F_t(\rho) > \overline{T}_i\right\}$$

$$= \{i \in N : \varphi_t \leq \underline{T}_i \text{ or } \varphi_t > \overline{T}_i\} = K_t.$$

Hence, $N \setminus (J_t \cup K_t) = M_t$ and in either case, $(F, J, K, M) = (F_t(\rho), J_t, K_t, M_t)$ satisfies (b).

To show that there is no other such quadruple, consider (F, J, K, M) satisfying (a)–(b). If $J \cup M = \emptyset$, then $J = J_0$, $M = M_0$ and (a) implies that $F = F_0(\rho) = \rho$. By (b) and $J = M = \emptyset$, $\rho_1 = \varphi_1 \leq F_0(\rho) = \rho$; as $\rho_{t+1} \leq \rho < \rho_t$, $t = 0$ and $(F, J, K, M) = (F_t(\rho), J_t, K_t, M_t)$. If $J \cup M \neq \emptyset$, then $\varphi_1 > F$. So, $\varphi_{q+1} \leq F < \varphi_q$ for a unique $q = 1, \ldots, |\Phi|$,

$$J = \left\{i \in N : \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} \geq B_i\right\} = \{i \in N : \underline{S}_i < F \leq \overline{S}_i \text{ and } B_i < R_i/4\}$$

$$= \{i \in N : \underline{S}_i < \varphi_q \leq \overline{S}_i \text{ and } B_i < R_i/4\} = J_q,$$

$$K = \left\{ i \in N : \frac{F(R_i\alpha_i - F)}{R_i\alpha_i^2} \le L_i \right\} = \{i \in N : F \le \underline{T}_i \text{ or } F > \overline{T}_i\}$$

$$= \{i \in N : \varphi_q \le \underline{T}_i \text{ or } \varphi_q > \overline{T}_i\} = K_q,$$

$$M = N \setminus (J \cup K) = N \setminus (J_q \cup K_q) = M_q,$$

and by (a), $F = F_q(\rho)$. Since $\varphi_{q+1} = F_q(\rho_{q+1}) \le F = F_q(\rho) < \varphi_q = F_q(\rho_q)$ and $F_q(\cdot)$ is strictly increasing, $\rho_{q+1} \le \rho < \rho_q$. As $\rho_{t+1} \le \rho < \rho_t$, it follows that necessarily $q = t$ and $(F, J, K, M) = (F_t(\rho), J_t, K_t, M_t)$.

(ii) immediately follows from (i) and Theorem 3.1, where substituting (10) into $U_i(x^*) = (\frac{R_i\alpha_i}{F} - 1)x_i^*$ establishes (25). □

Proof of Lemma 5.1 (i) Since the global utility function $U(\cdot)$ is continuous on the compact nonempty set $\{x \in \mathbb{R}^n : L \le x \le B\}$, $U(x)$ attains a maximum, say at $x^\star$. Given $v^\star := \alpha^T x^\star$, standard results in linear programming show that the problem of maximizing $\sum_{i=1}^n x_i(\frac{R_i\alpha_i}{v^\star+\rho} - 1)$ over $L \le x \le B$ with $\alpha^T x = v^\star$ admits an optimal solution $x^\#$ with $L_i < x_i^\# < B_i$ for at most one i; any such $x^\#$ has $U(x^\#) = U(x^\star)$ and is therefore globally optimal.

(ii) If $x_i^\star > L_i$ and $R_i\alpha_i \le \alpha^T x^\star + \rho$, then $(L_i, x_{-i}^\star)$ is feasible and

$$\begin{aligned} U(L_i, x_{-i}^\star) &= \sum_{j\in N\setminus\{i\}} x_j^\star \left(\frac{R_j\alpha_j}{\sum_{k\in N\setminus\{i\}} \alpha_k x_k^\star + \alpha_i L_i + \rho} - 1 \right) \\ &\quad + L_i \left(\frac{R_i\alpha_i}{\sum_{k\in N\setminus\{i\}} \alpha_k x_k^\star + \rho} - 1 \right) \\ &\ge \sum_{j\in N\setminus\{i\}} x_j^\star \left(\frac{R_j\alpha_j}{\alpha^T x^\star + \rho} - 1 \right) + L_i^\star \left(\frac{R_i\alpha_i}{\alpha^T x^\star + \rho} - 1 \right) \\ &\ge \sum_{j\in N\setminus\{i\}} x_j^\star \left(\frac{R_j\alpha_j}{\alpha^T x^\star + \rho} - 1 \right) + x_i^\star \left(\frac{R_i\alpha_i}{\alpha^T x^\star + \rho} - 1 \right) = U(x^\star); \end{aligned}$$

further, if either $x_{-i}^\star \ne 0$ or $R_i\alpha_i < \alpha^T x^\star + \rho$, then the inequality is strict, contradicting the optimality of $x^\star$. Next assume $x_{-i}^\star = 0$ and $R_i\alpha_i = \alpha^T x^\star + \rho = \alpha_i x_i^\star + \rho$. For $L_i < \delta < x_i^\star$ $(\le B_i)$, it then follows that $(\delta, x_{-i}^\star)$ is feasible, $\alpha_i\delta + \rho < \alpha_i x_i^\star + \rho = R_i\alpha_i$ and

$$U(\delta, x_{-i}^\star) = \delta \left(\frac{R_i\alpha_i}{\alpha_i\delta + \rho} - 1 \right) > 0 = U(x^\star),$$

contradicting the optimality of $x^\star$.

(iii) Let $x^\star$ be a globally optimal solution (existence follows from (i)). If $R_i\alpha_i \le \rho$ for all $i \in N$, then (ii) implies that $x_i^\star = L_i$ for all $i \in N$, i.e., $x^\star = L$. Next, if $L = 0$ and $R_i\alpha_i > \rho$ for some $i \in N$, then for $0 < \delta < \frac{R_i\alpha_i - \rho}{\alpha_i}$, x with $x_i = \delta$ and $x_{-i} = 0$ is feasible and has $U(x) = \delta(\frac{R_i\alpha_i}{\alpha_i\delta+\rho} - 1) > 0 = U(0)$, assuring that 0 is not globally optimal. □

Proof of Lemma 5.2 (i) For $t \in N$ and $v > \omega_{t-1}$,

$$\frac{d\widehat{U}^t(v)}{dv} = \frac{R_t\rho + \sum_{k=1}^{t-1}(R_t - R_k)\alpha_k B_k + \sum_{k=t}^{n}(R_t - R_k)\alpha_k L_k}{(v+\rho)^2} - \frac{1}{\alpha_t} \tag{55}$$

and

$$\frac{d^2\widehat{U}^t(v)}{dv^2} = \frac{-2[R_t\rho + \sum_{k=1}^{t-1}(R_t - R_k)\alpha_k B_k + \sum_{k=t}^{n}(R_t - R_k)\alpha_k L_k]}{(v+\rho)^3}. \tag{56}$$

If $R_t\rho + \sum_{k=1}^{t-1}(R_t - R_k)\alpha_k B_k + \sum_{k=t}^{n}(R_t - R_k)\alpha_k L_k \le 0$, then (55) is always negative and $\widehat{U}^t(v)$ is strictly decreasing; in the alternative case, (56) is always negative and $\widehat{U}^t(v)$ is strictly concave.

(ii) Consider $t \in N \setminus \{n\}$ and $v \ge \omega_t$. From (35) and (38),

$$\widehat{U}^t(v) - \widehat{U}^{t+1}(v) = \left[\left(\frac{R_t}{v+\rho} - \frac{1}{\alpha_t}\right) - \left(\frac{R_{t+1}}{v+\rho} - \frac{1}{\alpha_{t+1}}\right)\right](v - \omega_t) \ge 0,$$

with equality holding if and only if $v = \omega_t$. Further, differentiating this difference with respect to v and evaluating it at $v = \omega_t$ yields

$$\begin{aligned}&\frac{d\widehat{U}^t(\omega_t)}{dv} - \frac{d\widehat{U}^{t+1}(\omega_t)}{dv}\\ &\quad= \frac{R_{t+1} - R_t}{(\omega_t+\rho)^2}\cdot(\omega_t - \omega_t) + \left(\frac{R_t}{\omega_t+\rho} - \frac{1}{\alpha_t}\right) - \left(\frac{R_{t+1}}{\omega_t+\rho} - \frac{1}{\alpha_{t+1}}\right) > 0,\end{aligned}$$

where the inequality follows from (38).

(iii) Equation (34) implies that the maximization problem defining $U^\#(v)$ (by (37)) is an n-item continuous knapsack problem, where item $i \in N$ has unit value $a_i := \frac{R_i\alpha_i}{v^\star+\rho} - 1$, unit weight $c_i := \alpha_i$, lower bound L_i and availability B_i. The optimal solution of this problem is obtained by indexing the firms in decreasing order of $\frac{a_i}{c_i} = \frac{R_i}{v+\rho} - \frac{1}{\alpha_i}$, first allocating L_i to each $i \in N$ and then distributing $v - \alpha^T L$ in increasing of the firms' indices up to their upper bounds. Due to the strict inequalities in (38), for $v \in \mathcal{I}_t$, a unique optimal solution is $(B_1, \dots, B_{t-1}, L_t + \frac{v-\omega_{t-1}}{\alpha_t}, L_{t+1}, \dots, L_n)$ and (using (35) and (36)) $U^\#(v) = \widehat{U}^t(v) = \widehat{U}(v)$. Since this is true for all $t \in N$, $U^\# = \widehat{U}$ on $[\omega_0, \omega_n]$.

(iv) To show that for each $1 \le t \le n$, there exists a value $\omega_0 \le v^t \le \omega_t$ such that $\widehat{U}$ is strictly increasing on $[\omega_0, v^t]$ and strictly decreasing on $[v^t, \omega_t]$, use induction. This holds for $t = 1$, since $\widehat{U}(v) = \widehat{U}^1(v)$ on $[\omega_0, \omega_1]$ by (iii) and $\widehat{U}^1$ is unimodal by (i). Suppose that for $1 \le t-1 \le n-1$, $\widehat{U}(v)$ is strictly increasing on $[\omega_0, v^{t-1}]$ and strictly decreasing on $[v^{t-1}, \omega_{t-1}]$ for some $\omega_0 \le v^{t-1} \le \omega_{t-1}$. If $v^{t-1} = \omega_{t-1}$, then $\widehat{U}(v)$ is strictly increasing on $[\omega_0, v^t]$ and strictly decreasing on $[v^t, \omega_t]$ for some $\omega_0 \le v^t \le \omega_t$ (by (i)). Alternatively, if $v^{t-1} < \omega_{t-1}$, then $\frac{d\widehat{U}^{t-1}(\omega_{t-1})}{dv} < 0$, by (39), $\frac{d\widehat{U}^t(\omega_{t-1})}{dv} < 0$ and, therefore, by the unimodality of $\widehat{U}^t$, it must be strictly decreasing on $[\omega_{t-1}, \infty)$. This implies that $\widehat{U}$ is strictly increasing on $[\omega_0, v^{t-1}]$ and strictly decreasing on $[v^{t-1}, \omega_t]$.

(v) Evidently,

$$\max_{L\le x\le B} U(x) = \max_{\omega_0\le v\le\omega_n}\left[\max_{L\le x\le B, \alpha^T x = v} U(x)\right] = \max_{\omega_0\le v\le\omega_n} U^\#(v);$$

the restriction $\omega_0 \leq v \leq \omega_n$ can be imposed as $\{x \in \mathbb{R}^n : L \leq x \leq B, \alpha^T x = v\} \neq \emptyset$ if and only if $\omega_0 \leq v \leq \omega_n$. Let $v^\star \in \mathcal{I}_{t^\star}$ be the unique maximizer of $U^\#$ over $[\omega_0, \omega_n]$ (existence and uniqueness follow from (iv)). Then any maximizer of U over $X^\star := \{x \in \mathbb{R}^n : L \leq x \leq B, \alpha^T x = v^\star\}$ is a globally optimal solution and (iii) shows that $x^\star$ given by (40) is such a maximizer. Now suppose x' is a globally optimal solution. Then $U^\#(\alpha^T x') \geq U(x') \geq U(x^\star) = U^\#(v^\star)$ and the uniqueness of $v^\star$ implies that $\alpha^T x' = v^\star$ and that x' attains the maximum of $U(x)$ over $X^\star$; by (iii), $x^\star$ is the unique maximizer, hence $x' = x^\star$. □

Proof of Theorem 5.1 Let $t^\star$ and $v^\star$ be as in the statement of the theorem; both are well defined and unique, since $\widehat{U}^{t^\star}$ is unimodal (by Lemma 5.2(i)). From Lemma 5.2(iv), $U^\#$ is unimodal on $[\omega_0, \omega_n]$, so it has a unique maximizer on this interval, say v'. If $\omega_{t^\star-1} \leq v' \leq \omega_{t^\star}$, then $v' = v^\star$, since $\widehat{U}^{t^\star}$ has a unique maximizer. If $v' > \omega_{t^\star}$, then for some $k \geq 1$,

$$U^\#(v') = \widehat{U}^{t^\star+k}(v') \leq \widehat{U}^{t^\star}(v') < \widehat{U}^{t^\star}(v^\star) = U^\#(v^\star),$$

contradicting the optimality of v'. Finally, if $v' < \omega_{t^\star-1}$, then $t^\star > 1$. By definition of $t^\star$, $\widehat{U}^t$ is strictly increasing on $[\omega_{t-1}, \omega_t]$ for each $t < t^\star$, so $U^\# = \widehat{U}$ is strictly increasing on $[\omega_0, \omega_{t^\star-1}]$, again contradicting the optimality of v'. The conclusion that $x^\star$ is a unique globally optimal solution now follows from Lemma 5.2(v). Finally, observe that as $x_i^\star > L_i$ for $i \leq t^\star - 1$, Lemma 5.1(ii) assures that $R_i\alpha_i > \rho$, implying that $t^\star - 1 \leq |\{i \in N : R_i\alpha_i > \rho\}|$. □

Proof of Lemma 5.3 For $v \geq 0$, let $f_{ij}(v) := \frac{R_i - R_j}{v+\rho} + \frac{1}{\alpha_j} - \frac{1}{\alpha_i}$. Then $f'_{ij}(v) = \frac{R_j - R_i}{(v+\rho)^2}$ and, by (41), $f_{ij}(0) > 0$.

(i) If $R_i \leq R_j$, then $f'_{ij}(v) = \frac{R_j - R_i}{(v+\rho)^2} \geq 0$ for each $v \geq 0$, implying that $f_{ij}(v)$ is increasing in $v > 0$. As $f_{ij}(0) > 0$, conclude $f_{ij}(v) > 0$ for all $v \geq 0$, verifying (42). Next, if $R_i > R_j$ and $\alpha_i \geq \alpha_j$, then $f_{ij}(v) > 0$ for all $v \geq 0$, again verifying (42).

(ii) If $R_i > R_j$ and $\alpha_i < \alpha_j$, then $f'_{ij}(v) < 0$ for each $v \geq 0$, so $f_{ij}(v)$ is strictly decreasing in $v \geq 0$. Further, $v_{ij} = \frac{R_i - R_j}{\frac{1}{\alpha_i} - \frac{1}{\alpha_j}} - \rho > 0$ is the unique root of f_{ij}, where the positivity follows from (41). The conclusions (43)–(45) now follow easily. □

Proof of Corollary 5.1 Equation (46) implies that

$$\frac{R_1}{\rho} - \frac{1}{\alpha_1} > \cdots > \frac{R_n}{\rho} - \frac{1}{\alpha_n};$$

i.e., (41) holds for $1 \leq i < j \leq n$. Since (46) also implies the assumptions of Lemma 5.3(i) for $1 \leq i < j \leq n$, the corresponding conclusion of that lemma assures that (38) holds. Next, (47) implies that for $v \geq 0$, $\frac{R_i}{v+\rho} - \frac{1}{\alpha_i} = \frac{1}{\alpha_i}[\frac{R_i\alpha_i}{v+\rho} - 1]$ is strictly decreasing in $i \in N$, again verifying (38). If (48) holds, then (38) is trivially satisfied, as $\widehat{U}(v)$ is not defined for $v > \sum_{k=1}^n \alpha_k B_k$. Under any one of these three conditions, since (38) holds, the conclusions of Theorem 5.1 follow. □

References

1. Canbolat, P.G., Golany, B., Mund, I., Rothblum, U.G.: A stochastic competitive R&D race where "winner takes all". Oper. Res. **60**, 700–715 (2012). doi:10.1287/opre.1120.1055
2. Gerchak, Y., Parlar, M.: Allocating resources to research and development projects in a competitive environment. IIE Trans. **31**, 827–834 (1999)
3. Rosen, J.B.: Existence and uniqueness of equilibrium points for concave N-person games. Econometrica **33**, 520–534 (1965)
4. Veinott, A.F.: Unpublished Lecture Notes in Supply-Chain Optimization-MS&E 361, Stanford University, HW 3, Problem 3 and its solution (2000)
5. Milgrom, P., Roberts, J.: Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica **58**, 1255–1277 (1990)