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Multiple agents finitely repeated inspection game with dismissals

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Abstract This paper deals with an inspection game between a single inspector and several independent (potential) violators over a finite-time horizon. In each period, the inspector gets a renewable inspection resource, which cannot be saved and used in future periods. The inspector allocates it to inspect the (potential) violators. Each violator decides in each period whether to violate or not, and in what probability. A violation may be detected by the inspector with a known and positive probability. When a violation is detected, the responsible violator is "dismissed" from the game. The game terminates when all the violators are detected or when there are no more remaining periods. An efficient method to compute a Nash equilibrium for this game is developed, for any possible value of the (nominal) detection probability. The solution of the game shows that the violators always maintain their detection probability below 0.5.

Keywords Inspection games · Repeated games · Resource allocation · Nash equilibrium

1 Introduction

"Inspection games" are theoretic models used to describe real-world scenarios of a conflict between two or more asymmetric, opposing and strategic parties. Examples for such scenarios include arms control, auditing and accounting (insurance), and environmental regulatory enforcement. Usually, there is some kind of a hierarchical structure in these scenarios: An agency wishes to verify compliance from its subordinates by using inspections. Typically, the agency's resources are limited, so the verification can only be partial. As the agency and its subordinates are strategic, these scenarios should be modeled as non-cooperative games.

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Further, as most of these scenarios are repetitive by nature, with a finite horizon, they should be modeled as finitely repeated games.

Risk management has become a vital topic both in academia and practice during the past several decades. Most business intelligence tools have been used to enhance risk management, and the risk management tools have benefited from business intelligence approaches. This introductory article provides a review of the state-of-the-art research in business intelligence in risk management, and of the work that has been accepted for publication in this issue.

Inspection games deal with strategic risk: The risk of being manipulated, cheated, or violated intentionally. Risks, and managing different risks, which result from natural, political, economic, technical, and strategic sources have become an important subject nowadays, especially after recent traumatic events such as the 9/11/2001, and the subsequent anthrax attacks in the US (see, e.g., Wu et al. 2014 for some recent publications). In Bier (2011), the authors discuss approaches for applying game theory to the problem of defending against strategic attacks, and emphasize that "protecting against intentional attacks is fundamentally different from protecting against accidents or acts of nature" (see also Golany et al. 2009). In Heal and Kunreuther (2007), the authors study "interdependent risks problems" (risks whose effect depend on the player's own risk management strategies and on those of the others). They use game models to characterize their Nash equilibria, and to understand how individuals manage risks which are also affected by actions of others.

In the current paper we deal with strategic risk. In particular, we model and solve a non-cooperative finitely repeated inspection game between an inspector and multiple independent potential violators. We provide the inspector, who faces the risks of being violated, with the optimal way to use its limited resources in order to minimize the losses from violations. In order to have more deterrence power, if the inspector detects a violation, it penalizes the violator (to be denoted henceforth as the "agent") by "dismissing" it from the game. A concrete example is an insurance company who uses inspections against the well-known "moral hazard" phenomenon. Clearly, the "moral hazard" phenomenon has its price. Thus, the insurance company needs to determine the best inspection scheme to minimize or even to deter it (see, e.g, Borch 1982, 1990). Further, in our model, if the insurance company detects a violation, then the responsible agent will immediately lose its insurance and/or the possibility of future insurance. Another practical example is the Environmental Protection Agency that inspects manufacturing plants suspected of releasing toxic gases to the atmosphere, and punishes them in case of detection by forcing them to shut down plants that were caught with violations.

The game modeled in this paper is an extension of previous research (Deutsch et al. 2011, 2013). In Deutsch et al. (2011), the authors model a single-stage inspection game between a single inspector and several independent agents. The inspector has a finite resource for its inspections. It has to determine how to allocate it across the agents. Each one of the agent needs to decide whether to violate or not, and if it decides to violate, then with what probability. The authors developed an efficient method to find all Nash equilibria solutions for this game and discussed the special properties of their results.

In Deutsch et al. (2013), the authors consider a single-stage inspection game between a single inspector and a single agent with some known locations where the players can act. The inspector has a finite resource for its inspections. It has to decide how to divide its resource across the locations. In addition to the global resource restriction, it also has to adhere to some local restrictions: It cannot allocate to any location more than a specific amount of resource. The agent has a violation budget, which is some material resource. It has to decide if, where, and how to violate. Specifically, it has to decide how much of the material will be used for violation and where. All Nash equilibria solutions are found for this game, in an efficient method. Sensitivity analysis reveals some interesting phenomena of the equilibrium solutions, such as the non-monotonicity of the inspector's utility as a function of its inspection resource.

In the current paper, we model a non-cooperative finitely repeated inspection game between a single inspector and multiple independent agents. In this game, the inspector has a fixed renewable budget for its inspections. This budget cannot be carried over and used in subsequent periods. The inspector has to adhere to some local restrictions: It cannot allocate to any agent more than a specific amount. Each period, the inspector has to decide how to allocate the resource for inspection of the agents. In each period, each agent decides if and with what probability to violate. When an agent violates, there is a positive and known probability that its violation will be detected by the inspector, and then it will be dismissed from the game. Thus, the number and the identity of the agents may differ from one period to the next. The latter implies that this game is not a regular repeated game, where the same stage game repeats itself in each period. The game terminates when the preplanned number of periods is exhausted or when there are no more remaining agents. We develop an efficient and an explicit method to determine a Nash equilibrium for the game, which is applicable even in situations where the game's data is very large. In contrast, other ways for solving the game (e.g., dynamic programming formulations) may have a very high complexity. Our results show that under reasonable assumptions on the game's parameters, the agents always violate at equilibrium. In particular, they violate such that their detection probability will not exceed 0.5.

For an extended summary of previous literature on inspection games, we refer the reader to the literature reviews in Deutsch et al. (2011, 2013), or other literature surveys in this area (e.g., Avenhaus et al. 2002). Here, we will review some relevant and recent research done in the area of inspection games. In Dechenaux and Samuel (2012), the authors consider an infinitely repeated two person inspection game, with an existence of a third party; a regulator. The agent can offer a bribe to the inspector. The regulator does not make any strategic decisions, but it can inspect both players, and punish them in case of corruption. The authors characterize the set of bribes that can be sustained as equilibrium paths using the trigger strategy.

In Fukuyama et al. (1995), the authors consider an infinitely repeated two person game. A systematic long-term inspection strategy is introduced and is shown to induce compliance. As this policy alone can not induce full compliance, a supplementary penalty is added to achieve it. In Casas-Arce (2010), the author considers two models of a finitely repeated two person game. In the first model, the first player can decide to let the second player continue or to dismiss it and replace it by a new one. In the second model, the second player can also quit. The stage game is not changed due to replacements of the second player. The "outside" value for the second player is an exogenous factor. The author determines a Nash equilibrium for the first model, and this equilibrium fits the second model when the "outside" value for the second player is small.

In Bier and Haphuriwat (2011), the authors consider a game between a single inspector and a single or several agents, with the inspector being the first player. The authors identify the optimal proportion of containers that should be inspected by the inspector, in order to minimize its expected loss. Their results indicate that threatening to retaliate against violations may be beneficial to the inspector, as long as the threat is credible.

In Bakir (2011), the author considers a Stackelberg resource allocation game between a single defender and a single attacker, where the defender plays first. The defender allocates resources to secure sites against unauthorized weapon insertion, and then the attacker chooses a site to insert the weapon. The results in the basic model, with a single container route, suggest that in equilibrium the defender should maintain an equal level of security at each site, and that the level of resources needed increase as a function of the attacker's capability to detonate the weapon remotely. In the general model, with multiple container routes, the defender has some flexibility between securing foreign seaports and sites on the container route.

In Golany et al. (2012), the authors consider a game between a single defender and a single attacker. The defender allocates multiple defensive resources to protect multiple sites. The effectiveness of the resources in reducing damage vary across the resources and across the sites. The original game has piecewise linear utility functions and polyhedral action sets. The authors simplify the game, and solve a linearized version of it.

In Rothenstein and Zamir (2002), the authors consider a zero sum two person game, played over a fixed interval of time. The time after a violation is occurred and until its detection affects the payoffs. The inspector can inspect only once. The agent is not aware of the inspector's actions. The inspection can detect a real violation, fail to detect a real violation, and can also detect a violation which has not occurred. The authors determine the unique Nash equilibrium for this game.

The rest of the paper is organized as follows: Sect. 2 presents the formulation of the game. Section 3 provides an efficient method for determining a Nash equilibrium for the game. Section 4 illustrates the method using numerical examples. Summary of the results, conclusions, and possible directions for future research are presented in Sect. 5. All the proofs appear in the Appendix.

2 Model formulation

We consider a non-cooperative non-zero sum finitely repeated inspection game between a single inspector and several independent agents. The set of agents is denoted by $\mathcal{N} = \{1, \dots, n\}$, with $n \geq 2$, and the set of periods is denoted by $\mathcal{T} = \{1, \dots, T\}$. All the players enter the game in the first period. The inspector and the agents may have conflicting interests. When the inspector inspects a violating agent, the violation may be detected with some known and positive probability. A detected agent must leave the game as an additional penalty, and there is no utility for it in that period and beyond (but it does not lose its previous payoffs). We denote by $\mathcal{N}(t) \subseteq \mathcal{N}$ the set of *active agents* in period $t \in \mathcal{T}$, with $n(t) \equiv |\mathcal{N}(t)|$. We assume that there is complete information in the game. The game terminates when the preplanned number of periods is exhausted or when all the agents are detected.

The inspector has a fixed renewable periodic global budget $W > 0$, which cannot be carried over and used in subsequent periods. Its goal is to minimize its sum of losses from violations over all periods. In addition to the global budget constraint, it has some local constraints: The amount that it can allocate to each agent in each period is limited. So, in each period $t \in \mathcal{T}$ the inspector selects a vector from:

$$\begin{aligned} \mathcal{X}(t) \equiv \{x(t) \in \mathbb{R}^n : 0 \leq x_i(t) \leq \alpha_i \ \forall i \in \mathcal{N}(t), \\ x_i(t) = 0 \ \forall i \in \mathcal{N} \setminus \mathcal{N}(t), \text{ and } \sum_{i \in \mathcal{N}(t)} x_i(t) \leq W\}, \end{aligned} \tag{1}$$

where it is assumed that the inspector's allocations to all the non-active agents are 0. The set of allocations of the inspector throughout all the T periods is denoted:

$$\mathcal{X} \equiv \{x(t) \in \mathbb{R}^{n \times T} : x(t) \in \mathcal{X}(t) \ \forall t \in \mathcal{T}\}. \tag{2}$$

Each active agent wants to maximize its (expected) utility and to stay as long as possible in the game. Further, it is assumed that each agent has an incentive to violate in as many periods as possible. Hence, in each period where agent i is active, i.e., in each $t \in \mathcal{T}$ where $i \in \mathcal{N}(t)$, agent i selects a violation-probability from:

$$\mathcal{Y}_i(t) \equiv \{0 \le y_i(t) \le 1\}. \tag{3}$$

The set of violation-probabilities of all the active agents in period $t \in \mathcal{T}$ is denoted:

$$\mathcal{Y}(t) \equiv \mathcal{Y}_{i_1}(t) \times \cdots \times \mathcal{Y}_{i_{n(t)}}(t), \tag{4}$$

where $\mathcal{N}(t) \equiv \{i_1, \dots, i_{n(t)}\}$ (see the paragraph below (13)).

The set of violation-probabilities of agent $i \in \mathcal{N}$ in all its active periods is denoted:

$$\mathcal{Y}_i \equiv \{y_i(t) \in \mathbb{R}^{t \in \mathcal{T}: i \in \mathcal{N}(t)} : y_i(t) \in \mathcal{Y}_i(t)\} \tag{5}$$

In this model,there is no discounting of future payoffs. The utility functions of the inspector and of agent $i \in \mathcal{N}$ depend on selection of $x \in \mathcal{X}$ and $y_i \in \mathcal{Y}_i$ of the inspector and agent i, respectively, and are expressed by:

$$\hat{U}^I(x, \{y_i\}_{i \in \mathcal{N}}) \equiv \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}(t)} -y_i(t)(C_i - A_i x_i(t)), \tag{6}$$

and

$$U^i(x_i, y_i) \equiv \sum_{t \in \mathcal{T}: i \in \mathcal{N}(t)} y_i(t)(B_i - D_i x_i(t)) \tag{7}$$

with

$$A_i, B_i, C_i, D_i > 0 \quad \text{for all } i \in \mathcal{N}. \tag{8}$$

C_i represents the inspector's loss from violation of agent i, A_i represents the inspector's reduced loss from inspecting agent i, B_i represents agent i's payoff from violation, and D_i represents agent i's reduced payoff due to inspection. It is assumed that:

$$\alpha_i < \min\left\{\frac{B_i}{D_i}, \frac{C_i}{A_i}\right\} \quad \forall i \in \mathcal{N}, \tag{9}$$

the latter constraint expresses the assumption that the inspector's utility is non-positive and each agent i's utility is non-negative, no matter what amount of resource the inspector decides to invest in inspection of agent i. This assumption eliminates potential situations of "over protection".

A *Nash equilibrium* is a joint set of actions such that each player's action is *a best response* to the other players' actions, and formally:

$$\hat{U}^I(x^*, \{y_i^*\}_{i \in \mathcal{N}}) = \max_{x \in \mathcal{X}} \hat{U}^I(x, \{y_i^*\}_{i \in \mathcal{N}}), \tag{10}$$

and

$$U^i(x_i^*, y_i^*) = \max_{y_i \in \mathcal{Y}_i} U^i(x_i^*, y_i). \tag{11}$$

Nash equilibria with the inspector's utility function given by (6), with assumption (9) are invariant with respect to the C_i's. So, for simplicity, the inspector's utility function will be expressed from now on by:

$$U^I(x, \{y_i\}_{i \in \mathcal{N}}) \equiv \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}(t)} (A_i y_i(t)) x_i(t), \tag{12}$$

which represents its total "reduced loss" from violations (due to its inspections).

W.l.o.g, and in order to avoid discussing degenerate situations, it is assumed that:

$$A_1 > \cdots > A_n, \tag{13}$$

and recall that in each $t \in \mathcal{T}$ there can be only a subset of the original set of players. Hence, in each period $t \in \mathcal{T}$ we will reorder the agents and rename them such that $\mathcal{N}(t) \equiv \{i_1, \ldots, i_{n(t)}\}$, and:

$$A_{i_1} > \cdots > A_{i_{n(t)}}. \tag{14}$$

Further, we will define:

$$A_0 \equiv \infty, \tag{15}$$

and

$$A_{i_{n(t)+1}} \equiv 0. \tag{16}$$

If the inspector inspects agent $i \in \mathcal{N}(t)$ in period $t \in \mathcal{T}$ while it violates, there is a positive probability that its violation will be detected. The probability of detecting agent i's violation is expressed by:

$$P_i(t) \equiv \frac{x_i(t)y_i(t)\beta}{\alpha_i}. \tag{17}$$

Specifically, the inspection intensity of agent $i \in \mathcal{N}(t)$ in period $t \in \mathcal{T}$ is denoted by:

$$Q_i(t) \equiv \frac{x_i}{\alpha_i}, \tag{18}$$

and $\beta \in (0, 1]$ represents a nominal detection probability. So, if i agent is inspected with an inspection intensity of 1, and violates with probability 1, the probability that its violation will be detected is β. Hence, agent $i \in \mathcal{N}(t)$'s expected utility in period $t \in \mathcal{T}$ is:

$$E[U^i(t)] \equiv y_i(t)(B_i - D_i x_i(t))(1 - P_i(t)) = (B_i - D_i x_i(t))\left(y_i(t) - \frac{x_i(t)y_i(t)^2\beta}{\alpha_i}\right). \tag{19}$$

Now,

$$\frac{\partial E[U^i(t)]}{\partial y_i(t)} \equiv (B_i - D_i x_i(t))\left(1 - \frac{2x_i(t)y_i(t)\beta}{\alpha_i}\right) = 0 \Rightarrow y_i(t) = \frac{\alpha_i}{2\beta x_i(t)}, \tag{20}$$

and

$$\frac{\partial^2 E[U^i(t)]}{\partial (y_i(t))^2} \equiv (B_i - D_i x_i(t))\frac{(-2x_i(t)\beta)}{\alpha_i}, \tag{21}$$

which is negative if $0 < x_i(t) < \frac{B_i}{D_i}$. By assumption (9), $\alpha_i < \frac{B_i}{D_i}$ for all agents $i \in \mathcal{N}$, and by (1), $x_i(t) \le \alpha_i$ for all $i \in \mathcal{N}(t)$ and $t \in \mathcal{T}$. Hence, if $x_i(t) > 0$, this is a maximum point.

Finally, as each violation probability must be (trivially) less or equal to 1, when the allocation is $x_i(t) > 0$, the optimal violation-probability of agent $i \in \mathcal{N}(t)$ in period $t \in \mathcal{T}$ is given by:

$$y_i(t)^{opt\ for\ x_i(t)>0} = \min\left\{1, \frac{\alpha_i}{2\beta x_i(t)}\right\}. \tag{22}$$

As each agent wants to maximize its sum of expected utilities, it maximizes:

$$E[U^i(x_i, y_i)] \equiv \sum_{t \in \mathcal{T}: i \in \mathcal{N}(t)} E[U^i(x_i, y_i)(t)]. \tag{23}$$

Note that (22) represents agent $i \in \mathcal{N}(t)$'s best response to $x_i(t) > 0$ in period $t \in \mathcal{T}$. Further, if $x_i(t) = 0$, as there is no possibility of detection, and the expected utility is positive, the best response of agent $i \in \mathcal{N}(t)$ in period $t \in \mathcal{T}$ is:

$$y_i(t)^{opt\ for\ x_i(t)=0} = 1. \tag{24}$$

From now on, for convenience, we will write the utility functions without their domain. The next section introduces an efficient way to determine a Nash equilibrium for the game defined by: Decisions sets (1), (2), (3), (5), utilities (12), (19), (23) and assumptions (8), (9), (13).

3 Determining a Nash equilibrium

This section introduces a specific Nash equilibrium for the game. Throughout, the empty sum is defined to be 0, that is, for $t' < t$, $\sum_{u=t}^{t'} x(u) \equiv 0$. In this game $W(> 0)$ is fixed, renewable and cannot be carried over and used in later periods, so (without proof):

Corollary 1 *In each period $t \in \mathcal{T}$, the inspector uses* $\min\{W, \sum_{j \in \mathcal{N}(t)} \alpha_j\}$. □

Further, as all agents enter in the first period, and as they know that some of them might be detected and forced to leave, while W is fixed, no agent has an incentive to postpone its violation to a later period, that is, all players here are *myopic*.

We next introduce a theorem that determines a Nash equilibrium for a general stage game. Note that playing (the suitable) Nash equilibrium of the stage game in each period is a Nash equilibrium for the repeated game. Specifically, the theorem distinguishes between four different possible cases, which depend on the values of W, β, $\{A_i\}_{i \in \mathcal{N}}$, and specific conditions regarding them, and determines a Nash equilibrium for each possible case. Before introducing this theorem, the next lemma proves that exactly a single case of these four cases must exist.

Lemma 1 *There is always exactly one complete set of conditions of Scenario (i), or Scenario (ii) -Case (A), Case (B), or Case (C) that is satisfied.* □

Theorem 1 *Sufficient conditions for $(x(t)^*, y(t)^*) \in \mathcal{X}(t) \times \mathcal{Y}(t)$ to be a Nash equilibrium in period $t \in \mathcal{T}$ are listed below:*

(i) $\underline{W \geq \sum_{j \in \mathcal{N}(t)} \alpha_j}$*: Then,*

$$(x_i^*(t), y_i^*(t)) = \left(\alpha_i, \min\left\{1, \frac{1}{2\beta}\right\}\right) \text{ for each } i \in \mathcal{N} \tag{25}$$

(ii) $\underline{(\sum_{j=i_1}^{i_{k-1}} \alpha_j < W \leq \sum_{j=i_1}^{i_k} \alpha_j) \text{ for } k < n(t) \text{ or } (\sum_{j=i_1}^{i_{k-1}} \alpha_j < W < \sum_{j=i_1}^{i_k} \alpha_j) \text{ for } k = n(t)}$*:*

(A) If $\beta \leq \frac{1}{2}$, then,

$$(x_i^*(t), y_i^*(t)) = \begin{cases} (\alpha_i, 1) & \text{if } i = i_1, \ldots, i_{k-1}, \\ (W - \sum_{j=i_1}^{i_{k-1}} \alpha_j, 1) & \text{if } i = i_k, \\ (0, 1) & \text{if } i = i_{k+1}, \ldots, i_{n(t)}. \end{cases} \tag{26}$$

(B) For $u \in \{i_{k-1}, i_{k-2}, \ldots, i_1, 0\}$ *and* $m = i_k$*, if the following conditions hold:*

1. $\beta > \frac{1}{2}$,
2. $A_u \geq 2\beta A_m$,
3. $A_{u+1} \leq 2\beta A_m$,
4. $W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_m}$,
5. $W \geq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m-1} \alpha_j A_j}{2\beta A_m}$,

then,

$$(x_i^*(t), y_i^*(t)) = \begin{cases} \left(\alpha_i, \frac{1}{2\beta}\right) & \text{if } i = i_1, \ldots, u, \\ \left(\frac{\alpha_i A_i}{2\beta A_m}, \frac{A_m}{A_i}\right) & \text{if } i = u+1, \ldots, m-1, \\ \left(W - \sum_{j=i_1}^{m-1} x_j^*(t), 1\right) & \text{if } i = m, \\ (0, 1) & \text{if } i = m+1, \ldots, i_{n(t)}. \end{cases} \tag{27}$$

(C) For $u \in \{i_{k-1}, i_{k-2}, \ldots, i_1, 0\}$ *and* $m = i_k$*, if the following conditions hold:*

1. $\beta > \frac{1}{2}$,
2. $W \geq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{A_u}$,
3. $W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{A_{u+1}}$,
4. $W > \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_m}$,
5. $W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_{m+1}}$,

then, for $\lambda \equiv \sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta(W - \sum_{j=i_1}^{u} \alpha_j)}$,

$$(x_i^*(t), y_i^*(t)) = \begin{cases} \left(\alpha_i, \frac{1}{2\beta}\right) & \text{if } i = i_1, \ldots, u, \\ \left(\frac{\alpha_i A_i}{2\beta\lambda}, \frac{\lambda}{A_i}\right) & \text{if } i = u+1, \ldots, m, \\ (0, 1) & \text{if } i = m+1, \ldots, i_{n(t)}. \end{cases} \tag{28}$$

If $\beta > \frac{1}{2}$*, W fits Scenario (ii), and cases (B) or (C) are not satisfied with any u and* $m = i_k$*: Let* $m \leftarrow m+1$*, and check again the conditions of (B) and (C). This process can be repeated until* $m = i_{n(t)}$*.* □

We note that Theorem 1 continues to hold in situations where the inspector's renewable budget does not remain fixed from period to period (it still cannot be carried over and used in later periods). That is, even if the players know that the budget might be increased or decreased, but they don't know in advance which direction it will take, they stay myopic and the theorem continues to hold. The following corollary is given without proof, as it is an immediate result from Theorem 1. It lists all the detection probabilities of violations at equilibrium in period $t \in \mathcal{T}$.

Corollary 2 *The violation detection probabilities at equilibrium in period* $t \in \mathcal{T}$ *are listed below:*

(i) $\underline{W \geq \sum_{j \in \mathcal{N}(t)} \alpha_j}$:

$$P_i(t) = \begin{cases} \beta \left(\leq \frac{1}{2}\right) & \text{if } \beta \leq \frac{1}{2}, \forall i \in \mathcal{N}(t), \\ \frac{1}{2} & \text{if } \beta > \frac{1}{2}, \forall i \in \mathcal{N}(t). \end{cases} \tag{29}$$

(ii) $\underline{(\sum_{j=i_1}^{i_{k-1}} \alpha_j < W \leq \sum_{j=i_1}^{i_k} \alpha_j) \text{ for } k < n(t) \text{ or } (\sum_{j=i_1}^{i_{k-1}} \alpha_j < W < \sum_{j=i_1}^{i_k} \alpha_j) \text{ for } k = n(t):}$

(A)

$$P_i(t) = \begin{cases} \beta \left(\leq \frac{1}{2}\right) & \text{for } i = i_1, \ldots, i_{k-1} \\ \frac{x^*_{i_k}\beta}{\alpha_{i_k}} \left(\leq \frac{1}{2}\right) & \text{if } i = i_k, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

(B)

$$P_i(t) = \begin{cases} \frac{1}{2} & \text{for } i = i_1, \ldots, m-1 \\ \frac{x^*_m\beta}{\alpha_m} \left(\leq \frac{1}{2}\right) & \text{if } i = m, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

(C)

$$P_i(t) = \begin{cases} \frac{1}{2} & \text{for } i = i_1, \ldots, m \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

□

This corollary implies that in all circumstances, the detection probabilities at equilibrium are smaller than or equal to 0.5. This result may lead to future analysis of similar games where the agents' original utilities do not necessarily reflect risk neutral attitude of the players. For example, we conjecture that for strictly risk averse agents, the corresponding corollary would lead to detection probabilities at equilibrium that would be strictly smaller than 0.5.

4 A numerical example

The next example demonstrates the solution. Then, we show how the players' utilities in a single period change as a function of β.

Example 1 Consider the following example. There are four agents, three periods, $\beta = \frac{4}{5}$, $W = 400$, and

i	A_i	B_i	C_i	D_i	α_i
1	100	2,000	1,000,000	10	100
2	50	1,800	1,000,000	10	150
3	20	2,100	1,000,000	10	200
4	5	3,000	1,000,000	10	100

where the agents are ordered as in (13) and for all $i \in \mathcal{N}$, each α_i satisfies (9).

Period 1: In period 1, $\mathcal{N}(1) = \{1, 2, 3, 4\}$, and agents i_1, i_2, i_3, i_4 are agents $1, 2, 3, 4$, respectively. As $250 = \sum_{j=1}^{2} \alpha_j < W = 400 < \sum_{j=1}^{3} \alpha_j = 450$, Scenario (ii) is relevant, with $k = 3$. Since $\beta > \frac{1}{2}$, either Case (B) or Case (C) must hold. Consider Case (B) with $u = i_{k-1} = i_2$ and $m = i_k = i_3$ (agents 2,3, respectively). Checking its conditions (Condition 5 of Case (B) is always satisfied when $u = i_{k-1}, m = i_k$ by the range of W, so it will not be checked):

1. $\beta > \frac{1}{2}$,
2. $50 = A_2 \geq 2\beta A_3 = 32$,
3. $20 = A_3 \leq 2\beta A_3 = 32$, but
4. $400 = W > \alpha_1 + \alpha_2 + \frac{\alpha_3}{2\beta} = 375$,

That is, Condition 4 of Case (B) is not satisfied. Consider Case (C) with $u = i_{k-1} = i_2$ and $m = i_k = i_3$. Checking its conditions:

1. $\beta > \frac{1}{2}$,
2. $400 = W \geq \alpha_1 + \alpha_2 + \frac{\alpha_3 A_3}{A_2} = 330$,
3. $400 = W \leq \alpha_1 + \alpha_2 + \alpha_3 = 450$,
4. $400 = W > \alpha_1 + \alpha_2 + \frac{\alpha_3}{2\beta} = 375$,
5. $400 = W \leq \alpha_1 + \alpha_2 + \frac{\alpha_3 A_3}{2\beta A_4} = 750$.

So, in period 1, Nash equilibrium is achieved using Case (C) with $u = i_{k-1} = i_2, m = i_k = i_3$, and

$$(x_i^*(1), y_i^*(1)) = \begin{cases} \left(\alpha_i, \frac{1}{2\beta} = 0.625\right) & \text{for } i = 1, 2, \\ (W - \alpha_1 - \alpha_2, 0.833) & \text{for } i = 3, \\ (0, 1) & \text{for } i = 4. \end{cases} \tag{33}$$

The agents' expected utilities in this period are: $E[U^1(1)] = 312.5, E[U^2(1)] = 93.75, E[U^3(1)] = 250, E[U^4(1)] = 3000$. The inspector's (reduced) utility is $U^I(1) = 13437.5$.

Period 2: Suppose that agent 1's violation was detected in period 1 (by Corollary 2, the probability for it was $P_1(1) = 0.5$). So, $\mathcal{N}(2) = \{2, 3, 4\}$. In this period, agents i_1, i_2, i_3 are agents 2, 3, 4, respectively. As $350 = \sum_{j=i_1}^{i_2} \alpha_j < W = 400 < \sum_{j=i_1}^{i_3} \alpha_j = 450$, Scenario (ii) is relevant, with $k = 3$. Since $\beta > \frac{1}{2}$, either Case (B) or Case (C) must hold. Consider Case (B) with $u = i_{k-1} = i_2$ and $m = i_k = i_3$ (agents 3,4, respectively). Checking its conditions:

1. $\beta > \frac{1}{2}$,
2. $20 = A_3 \geq 2\beta A_4 = 8$,
3. $5 = A_4 \leq 2\beta A_4 = 8$,
4. $400 = W \leq \alpha_2 + \alpha_3 + \frac{\alpha_4}{2\beta} = 412.5$.

So, in period 2, Nash equilibrium is achieved using Case (B) with $u = i_{k-1} = i_2, m = i_k = i_3$, and

$$(x_i^*(2), y_i^*(2)) = \begin{cases} \left(\alpha_i, \frac{1}{2\beta} = 0.625\right) & \text{for } i = 2, 3, \\ (W - \alpha_1 - \alpha_2, 1) & \text{for } i = 4. \end{cases} \tag{34}$$

The agents' expected utilities in this period are: $E[U^2(2)] = 93.75, E[U^3(2)] = 31.25, E[U^4(2)] = 1500$. The inspector's utility is $U^I(1) = 7437.5$.

Period 3: Suppose that agent 4's violation was detected in period 2 (by Corollary 2, the probability for it was $P_4(2) = 0.4$). So, $\mathcal{N}(3) = \{2, 3\}$. In this period agents i_1, i_2 are agents 2, 3, respectively. As $W = 400 > \sum_{j=i_1}^{i_2} \alpha_j = 350$, Scenario (i) is relevant, with $k = 2$. So,

$$(x_i^*(3), y_i^*(3)) = \left\{ \left(\alpha_i, \min\left\{1, \frac{1}{2\beta}\right\} = 0.625\right) \text{ for } i = 2, 3. \right. \tag{35}$$

The agents' expected utilities in this period are: $E[U^2(3)] = 93.75$, $E[U^3(3)] = 31.25$. The inspector's utility is $U^I(3) = 7187.5$.

The agents' total expected utilities are $E(U^1) = 312.5, E(U^2) = 281.25, E(U^3) = 312.5, E(U^4) = 4500$. The inspector's total reduced utility is $U^I = 28062.5$.

Discussion: Although the numerical example is rather simple, we can see that the inspector's reduced loss decreases over time, as there are fewer agents and so it can inspect all of them. The agents' payoffs do not increase from one period to the other. If the inspector allocates the maximum possible amount of inspection resource to inspecting agent i (α_i), and detects no violation, then the agent's payoff in the subsequent period will be the same. If an agent is not inspected by the full amount of inspection resources associated with it, then its payoff in subsequent periods may be lower. So, the agents have no incentive to postpone their violations to later periods and hence they act myopically. Also, it is clear from the example, that agents decrease their violation probabilities as the amount of resource the inspector allocates for their inspection increases (e.g., the violation probabilities of agent 3 in periods 1 and 2).

Returning to period 1, that is, $\mathcal{N}(1) = \mathcal{N}$, let β change from 0 to 1. As $W = 400$, Scenario (ii) is relevant, with $k = 3$. For $\beta \in [0, \frac{1}{2}]$, Case (A) holds, and the Nash equilibrium by Theorem 1 is:

$$(x_i^*(1), y_i^*(1)) = \begin{cases} \left(\alpha_i, \min\left\{\frac{1}{2\beta}, 1\right\}\right) & \text{for } i = 1, 2, \\ (W - \alpha_1 - \alpha_2, 1) & \text{for } i = 3, \\ (0, 1) & \text{for } i = 4, \end{cases} \tag{36}$$

(note that as $\beta \in [0, \frac{1}{2}]$, $\min\{\frac{1}{2\beta}, 1\} = 1$. Thus, writing $y_i^*(1) = \min\{\frac{1}{2\beta}, 1\}$ for $i = 1, 2$ is exactly the same as writing $y_i^*(1) = 1$ for them, as is written in Case (A)). For $\beta \in (\frac{1}{2}, 1]$, either Case (B) or Case (C) must hold. Specifically, when $\beta \in (\frac{1}{2}, \frac{2}{3}]$, Case (B) holds with $u = 2, m = 3$, and when $\beta \in (\frac{2}{3}, 1]$, Case (C) holds with $u = 2, m = 3$. Now, for all these β values, the Nash equilibrium is given by (36). That is, the strategies of the inspector and of agents 3 and 4 in this example are not affected by the value of β. The strategies of agents 1 and 2 are expressed the same: $y_i^*(1) = \min\{\frac{1}{2\beta}, 1\}$ for $i = 1, 2$. But, clearly, for $\beta \in [0, \frac{1}{2}]$, $\min\{\frac{1}{2\beta}, 1\} = 1$, and for $\beta \in (\frac{1}{2}, 1]$, $\frac{1}{2} \leq \min\{\frac{1}{2\beta}, 1\} < 1$. That is, as β increases the violation probabilities of agents 1 and 2 decrease.

The agent's utilities as a function of $\beta \in [0, 1]$ are depicted in Fig. 1. As we expected, the agents' utilities decrease when β increases. Note that for $\beta \in [0, 0.5]$, all agents violate with probability 1, even if they are inspected by the full amount of inspection resources associated with them, as the probability that their violation will be detected is rather low.

The inspector's (reduced) utility for this period, and for the given W is depicted in Fig. 2. That is, for $\beta \in [0, 0.5]$ the inspector is indifferent about β's value. For $\beta \in (0.5, 1]$ the inspector's utility decreases when β increases.

5 Concluding remarks

This paper considers a finitely repeated non-cooperative non-zero sum inspection game between a single inspector and multiple potential violators. The inspector has a global and local budget constraints. It has to determine how to allocate its global budget among the potential violators, adhering to the local constraints, in order to detect violations. The potential violators are independent of each another. Each one has to choose in each period a violation probability, where there is a positive probability that its violation will be detected, and then it will be forced to leave the game.

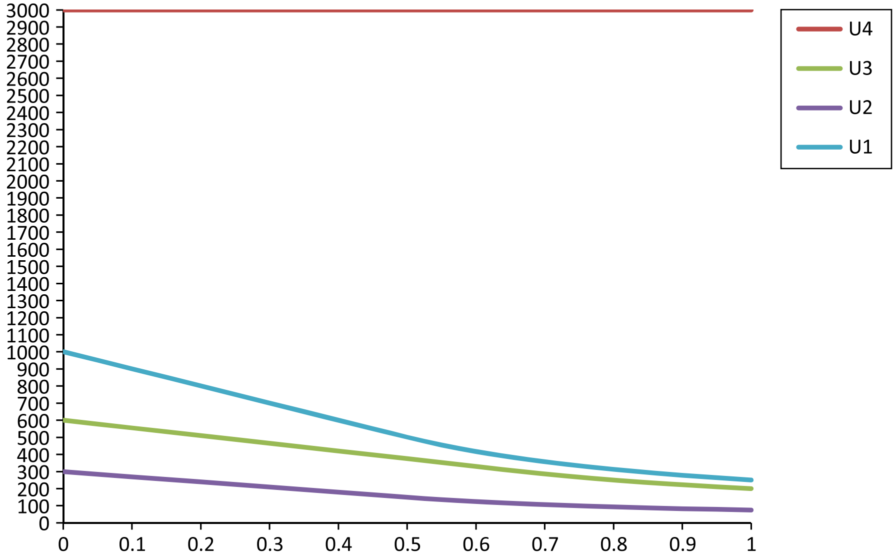

Fig. 1 The agents' expected equilibrium utilities as a function of β

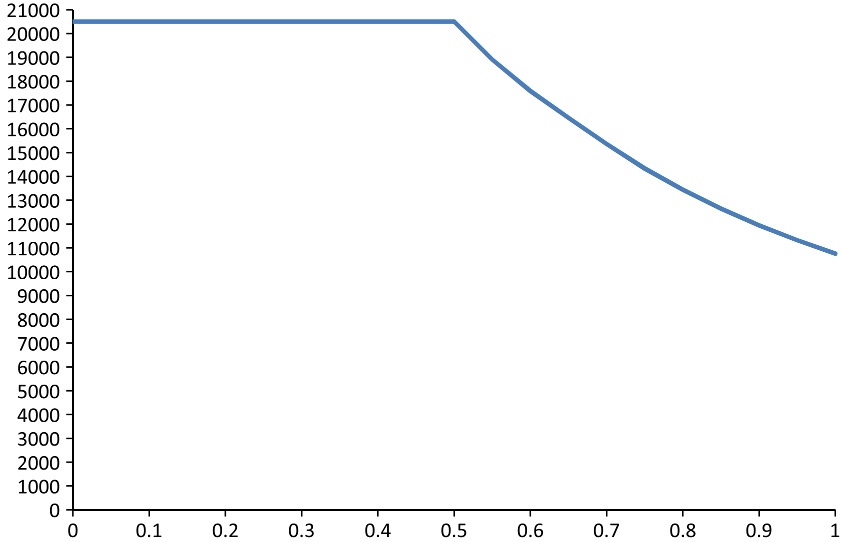

Fig. 2 The inspector's equilibrium (reduced) utility as a function of β

We introduce an efficient way to determine a Nash equilibrium for this game, parametrically in the inspector's global budget. Our results indicate that there are situations where the players choose the same strategy even when the values of beta are changed (see the numerical example). Further, even though the original utilities of the agents are bilinear, the introduction of detection probabilities causes them to become concave (19–21). Hence, the agents can be viewed as though they are risk averse, i.e., they prefer smaller guaranteed payoffs over larger but higher-risk payoffs. Their "risk aversion" is established by Corollary 2 that states that the agents violate such that their detection probabilities do not exceed 0.5.

There are many possibilities for further research. Some of which are pointed out below:

1. This paper assumes that all agents enter the game in the same period. In a follow-up work, which is now under progress, we investigate what happens when the agents do not necessarily all enter in the same period.
2. The agents in this paper are characterized by their myopic behavior. Their behavior may change if, for example, there is a dynamic sequence of new agents that enter the game in different periods. In such cases, agents that know that the number of agents in subsequent periods will increase, might prefer to postpone their violations, and will not be myopic.
3. In this paper, agents are assumed to be independent. Allowing them to cooperate with each other will open the door to completely new game structures with interesting implications.
4. The game modeled and solved in this paper can be applied to other areas of risk management, e.g., financial risks. Wu and Olson (2010) demonstrates risk management through validation of predictive scorecards for a bank. This bank developed a model to assess account creditworthiness. The model is validated and compared to credit bureau scores. Our approach assumes that certain agents may intentionally cheat, so that the validation scores may be higher than the actual ones. A suitable inspection policy can be of great help in this, as well as in other similar scenarios.

Appendix

Proof of Lemma 1: If $W \geq \sum_{j \in \mathcal{N}(t)} \alpha_j$, Scenario (i) is relevant, and the empty set of conditions of Scenario (i) is satisfied.

Otherwise, Scenario (ii) is relevant. Suppose that $\mathcal{N}(t) = \{i_1, \ldots, i_k\}$, so $m = i_k (= i_{n(t)})$ and is fixed (following, we will also prove this for $\mathcal{N}(t) = \{i_1, \ldots, i_k, \ldots, i_{n(t)}\}$).

If $\beta \leq \frac{1}{2}$, the single condition of Case (A) is satisfied. Otherwise, $\beta > \frac{1}{2}$, i.e., Condition 1 of Case (B) and of Case (C) is satisfied. We will show that there is a $u \in \{i_{k-1}, i_{k-2}, \ldots, i_1, 0\}$, such that also conditions 2–5 of Case (B) or conditions 2–5 of Case (C) are satisfied.
Consider Case (B):

We will show that there is always a $u \in \{i_{k-1}, i_{k-2}, \ldots, i_1, 0\}$, such that Condition 2 and Condition 3 of Case (B) are satisfied. According to (14)–(16),

$$\infty \equiv A_0 > A_{i_1} > \cdots > A_{i_k} (= A_{i_{n(t)}}) > A_{i_{k+1}} \equiv 0.$$

If for all $v \in \{i_{k-1}, \ldots, i_1, 0\}$, $A_v > 2\beta A_{i_k}$, then $u = i_{k-1}$ satisfies the conditions, as $A_{u+1} = A_{i_k} < 2\beta A_{i_k}$ because $\beta > \frac{1}{2}$. Otherwise, if for all $v \in \{i_{k-1}, i_{k-2}, \ldots, i_1\}$, $A_v < 2\beta A_{i_k}$, then $u = 0$ satisfies the conditions, as $\infty \equiv A_0 > 2\beta A_{i_k}$. Otherwise, there must be a $u \in \{i_{k-1}, \ldots, i_1, 0\}$ such that Conditions 2 and Condition 3 of Case (B) are satisfied. Now, using this u, if also Condition 4 is satisfied, then we are done.[1]

Otherwise, if for the specific u Condition 4 of Case (B) is not satisfied; Consider Case (C) with the same u. Clearly, Condition 4 of Case (C) is satisfied as it is the negation of Condition 4 of Case (B). Also, using the existence of Condition 4 of (C) and Condition 2 of (B); Condition 2 of (C) is satisfied, as $W > \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{2\beta A_{i_k}} \geq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{A_u}$. If also

[1] Condition 5 of Case (B) is redundant when $m = i_k$, because of the condition of W and using Condition 3 of Case (B) and (14), as $W > \sum_{j=i_1}^{i_{k-1}} \alpha_j \geq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{i_{k-1}} \alpha_j A_j}{2\beta A_{i_k}}$.

Condition 3 is satisfied, then all the conditions of Case (C) are satisfied with this u,[2] and the proof is completed.

Otherwise, increase u into $u' \leftarrow u+1$, and consider Case (C) with u'. Condition 2 of Case (C) with u' is satisfied as it is the negation of Condition 3 of Case (C) with u. Condition 4 of Case (C) with u' is satisfied using the existence of Condition 2 of Case (C) with u' and Condition 3 of Case (B) with u. So, if also Condition 3 of Case (C) is satisfied, then all the conditions of Case (C) are satisfied with this u', and the proof is completed, Case (C) holds for this u'.

Otherwise, consider Case (C) with $u'' \leftarrow u'+1$. Again, condition 2,4, (and 5) will be satisfied from the same reasons as before, and the only possible obstacle can be Condition 3. This process can be repeated until the point where Case (C) is considered with i_{k-1}. Under this substitution, Condition 3 of (C) is always satisfied, as it becomes $W \leq \sum_{j=i_1}^{i_k} \alpha_j$, which W clearly satisfies. Hence, when $m = i_k (= i_{n(t)})$ and $\beta > \frac{1}{2}$, there is a a $u \in \{i_{k-1}, i_{k-2}, \ldots, i_1, 0\}$ such that all the conditions of Case (B) or of Case (C) are satisfied.

Now, suppose that $\mathcal{N}(t) = \{i_1, \ldots, i_k, \ldots, i_{n(t)}\}$. That is, m can be increased. If $\beta \leq \frac{1}{2}$, the single condition of Case (A) is satisfied. Otherwise, $\beta > \frac{1}{2}$, i.e., Condition 1 of Case (B) and of Case (C) is satisfied. Consider Case (B) with $m = i_k$. We already showed that for $m = i_k$, there is always a $u \in \{i_{k-1}, i_{k-2}, \ldots, i_1, 0\}$, such that conditions 2–3 of Case (B) are satisfied. Also, Condition 5 of (B) is always satisfied when $m = i_k$ (see footnote 3). So, if also Condition 4 of (B) is satisfied, then we are done. Else, consider Case (C) with the same u and with $m = i_k$. As we showed above, conditions 4 and 2 are immediate. So, if also conditions 3 and 5 of Case (C) are satisfied, then the proof is completed; all the conditions of Case (C) with this u and with $m = i_k$ are satisfied. Else, there are three options:

1. Condition 3 is satisfied, Condition 5 is not.
2. Condition 5 is satisfied, Condition 3 is not.
3. Both conditions are not satisfied.

(1) **Condition 3 is satisfied, Condition 5 is not:**

In this option, the existence of Condition 3 of Case (C), i.e., $W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{A_{u+1}}$, and the negation of Condition 5 of Case (C), i.e., $W > \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$, imply that $A_{u+1} < 2\beta A_{i_{k+1}}$.

As with this u and with $m = i_k$ Condition 2 of Case (B) is satisfied, that is, $A_u \geq 2\beta A_{i_k}$, and as $2\beta A_{i_k} > 2\beta A_{i_{k+1}}$ by (14), conditions 2 and 3 of Case (B) are satisfied with this u and with $m = i_{k+1}$.

Hence, consider Case (B) with this u and with $m = i_{k+1}$. Condition 5 of Case (B) is satisfied as it exactly the negation of Condition 5 of Case (C) with this u and with $m = i_k$. If also Condition 4 of Case (B) is satisfied, then the proof is completed; all the conditions of Case (B) with this u and with $m = i_{k+1}$ are satisfied.

Otherwise, consider Case (C) with this u and with $m = i_{k+1}$. Again, conditions 4 and 2 are immediate. Conditions 3 is also immediate as Condition 3 of Case (C) with the same u and with $m = i_k$ is satisfied, and now the right term of Condition 3 is only increased. So, if also Condition 5 of Case (C) is satisfied, then the proof is completed; all the conditions of Case (C) with this u and with $m = i_{k+1}$ are satisfied.

Otherwise, again, from the existence of Condition 3 of Case (C) and the non-existence of Condition 5 of Case (C), we can deduce that conditions 2 and 3 of Case (B) are satisfied with

[2] Condition 5 of Case (C) is redundant when $m = i_{n(t)}$, as $W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}} \rightarrow \infty$.

the same u and with $m = i_{k+2}$, and we consider Case (B) with this u and with $m = i_{k+2}$. Continuing in this process, eventually, all the conditions of Case (B) will be satisfied, as the right term of Condition 4 of Case (B) increases with the increment of m, or all the conditions of (C) will be satisfied, as the right term of Condition 5 of Case (C) increases with the increment of m, and when $m = i_{n(t)}$, Condition 5 of (C) is always satisfied (see footnote 4).
(2) **Condition 5 is satisfied, Condition 3 is not:**

In this option, from the restriction on W, i.e., $W \leq \sum_{j=i_1}^{i_k} \alpha_j$, and from the existence of Condition 5 of Case (C), we cannot know for sure whether

(I) $\frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}} < \sum_{j=u+1}^{i_k} \alpha_j$, or

(II) $\sum_{j=u+1}^{i_k} \alpha_j < \frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$ (the equal sign fits both options, of course).

Suppose that (I) is the right one:

So, as $\frac{\sum_{j=u+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}} < \sum_{j=u+1}^{i_k} \alpha_j$, there must be indexes $j \in \{u+1, \ldots, i_k\}$ such that $A_j \leq 2\beta A_{i_{k+1}}$. Further, from the existence of Condition 5 of Case (C) with u and with $m = i_k$ and the negation of Condition 3 of Case (C), $A_{u+1} > 2\beta A_{i_{k+1}}$, so index $u+1$ is not one of them. Hence, let $v \in \{u+1, \ldots, i_{k-1}\}$ be the smallest index such that $A_v \geq 2\beta A_{i_{k+1}} \geq A_{v+1}$ (note that $v \geq u+1$), and consider Case (B) with v and with $m = i_{k+1}$. Conditions 2–3 are (of course) satisfied. Condition 4 of Case (B) is: $W \leq^? \sum_{j=i_1}^{v} \alpha_j + \frac{\sum_{j=v+1}^{i_{k+1}} \alpha_j A_j}{2\beta A_{i_{k+1}}}$. Condition 5 of Case (B) is: $W \geq^? \sum_{j=i_1}^{v} \alpha_j + \frac{\sum_{j=v+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$. If conditions 4 and 5 are satisfied, then the proof is completed; all the conditions of Case (B) with this v and with $m = i_{k+1}$ are satisfied. Otherwise, note that if Condition 5 is not satisfied, then Condition 4 is satisfied, so there are two options:

(A2) Condition 5 is not satisfied, Condition 4 is satisfied.
(B2) Condition 4 is not satisfied, Condition 5 is satisfied.

(A2) Condition 5 is not satisfied, Condition 4 is satisfied:

So, $W \leq \sum_{j=i_1}^{v} \alpha_j + \frac{\sum_{j=v+1}^{i_{k-1}} \alpha_j A_j}{2\beta A_{i_{k+1}}}$.

Consider Case (C) with $\bar{u} \leftarrow u+1$ and with $m = i_k$. Condition 2 is satisfied as it exactly the negation of Condition 3 of Case (C) with u and with $m = i_k$. Condition 4 is satisfied using Condition 2 of Case (C) with $\bar{u}$ and with $m = i_k$, and Condition 3 of Case (B) with u and with $m = i_k$. Condition 5 is satisfied using the existence Condition 5 of Case (B) with v and with $m = i_{k+1}$, because $u+1 \leq v$ and Condition 5 of (C): $W \leq^? \sum_{j=i_1}^{u+1} \alpha_j + \frac{\sum_{j=u+2}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$. As $W \leq \sum_{j=i_1}^{v} \alpha_j + \frac{\sum_{j=v+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$, suppose negatively that Condition 5 of Case (C) with $\bar{u}$ and with $m = i_k$ is not satisfied, that is, $W > \sum_{j=i_1}^{u+1} \alpha_j + \frac{\sum_{j=u+2}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$, implying that: $\sum_{j=i_1}^{u+1} \alpha_j + \frac{\sum_{j=u+2}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}} < \sum_{j=i_1}^{v} \alpha_j + \frac{\sum_{j=v+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$, a contradiction. So, if also Condition 3 of Case (C) is satisfied, then the proof is completed; all the conditions of Case (C) with $\bar{u}$ and with $m = i_k$ are satisfied.

Otherwise, consider Case (C) with $m = i_k$, and with $\tilde{u} \leftarrow \bar{u} + 1 (= u+2)$. Continuing in this process, eventually, Case (C) will hold for some u^*, as Condition 3 of Case (C) is always satisfied for $u^* = i_{k-1}$ and $m = i_k$.

(B2) Condition 4 is not satisfied, Condition 5 is satisfied:

So, $W > \sum_{j=i_1}^{v} \alpha_j + \frac{\sum_{j=v+1}^{i_{k+1}} \alpha_j A_j}{2\beta A_{i_{k+1}}}$.

Consider Case (C) with $\bar{u} \leftarrow u + 1$ and with $m = i_k$. Condition 2 is satisfied as it exactly the negation of Condition 3 of Case (C) with u and $m = i_k$. Condition 4 is satisfied using Condition 2 of (C) with $\bar{u}$ and with $m = i_k$, and Condition 3 of Case (B) with u and $m = i_k$. Condition 5 is satisfied using the existence of Condition 5 of Case (B) with v and $m = i_{k+1}$, because $u + 1 \le v$ and Condition 5 of (C): $W \le^{?} \sum_{j=i_1}^{u+1} \alpha_j + \frac{\sum_{j=u+2}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$, and $\sum_{j=i_1}^{u+1} \alpha_j + \frac{\sum_{j=u+2}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}} > \sum_{j=i_1}^{v} \alpha_j + \frac{\sum_{j=v+1}^{i_k} \alpha_j A_j}{2\beta A_{i_{k+1}}}$. So, if also Condition 3 of (C) is satisfied, then the proof is completed; all the conditions of Case (C) with $\bar{u}$ and with $m = i_k$ are satisfied.

Otherwise, consider Case (C) with $m = i_k$ and with $\tilde{u} = \bar{u} + 1 = u + 2$. Continuing in this process, eventually, Case (C) will hold for some u^*, as Condition 3 of (C) is always satisfied for $u^* = i_{k-1}$ and $m = i_k$.

Suppose that (II) is the right one:

So, $\frac{A_j}{2\beta A_{i_{k+1}}} > 1$ for $j = u + 1, \ldots, i_k$ and there cannot be an index u and $m = i_{k+1}$ such that conditions 2–3 of Case (B) are satisfied. Consider Case (C) with $u' \leftarrow u + 1$ and with $m = i_k$. Condition 2 is satisfied as it is the negation of Condition 3 of Case (C) with u and with $m = i_k$. Condition 4 is satisfied using the existence of Condition 3 of Case (B) and of Case (C) with u and with $m = i_k$. Condition 5 is satisfied using the restriction on W and the fact that $\frac{A_j}{2\beta A_{i_{k+1}}} > 1$ for $j = u+2, \ldots, i_k$. So, if also Condition 3 is satisfied, then the proof is completed; all the conditions of Case (C) with $u' \leftarrow u + 1$ and with $m = i_k$ are satisfied.

Otherwise, consider Case (C) with $u'' \leftarrow u' + 1 (= u + 2)$ and with $m = i_k$. Continuing in this process, eventually, all the conditions of Case (C) will be satisfied, as the right term of Condition 3 of (C) increases with the increment of u, and as for $u = i_{k-1}$, Condition 3 of (C) is always satisfied as it becomes: $W \le \sum_{j=i_1}^{i_k} \alpha_j$, which is the restriction on W.

(3) **Conditions 3 and 5 are not satisfied:**

Consider Case (C) with $v \leftarrow u + 1$ and with $m = i_k$. Then, Condition 2 of Case (C) is satisfied as it is the negation of Condition 3 of Case (C) with u and with $m = i_k$. Condition 4 is satisfied using the existence of Condition 2 of Case (C) with $v = u + 1$ and with $m = i_k$, and using Condition 3 of Case (B) with u and with $m = i_k$. Hence, we are left with conditions 3 and 5 of Case (C). If also they are satisfied, then the proof is completed; all the conditions of Case (C) with v and with $m = i_k$ are satisfied.

Otherwise, if Condition 3 is satisfied and Condition 5 is not, return to (1) with v and with $m = i_k$. Otherwise, if Condition 5 is satisfied and Condition 3 is not, return to (2) with v and with $m = i_k$. Otherwise, if both are not satisfied, then consider Case (C) with $v' \leftarrow v + 1 (= u + 2)$ and with $m = i_k$. Continuing in this process, if for all $\hat{v} \in \{u + 2, \ldots, i_{k-2}\}$ both conditions are not satisfied, then for $\tilde{v} = i_{k-1}$, Condition 3 is satisfied. So, if also Condition 5 is satisfied, then the proof is completed; all the conditions of Case (C) with $\tilde{v}$ and with $m = i_k$ are satisfied. Otherwise, if Condition 5 of Case (C) is not satisfied for $\tilde{v}$ and with $m = i_k$, then together with the restriction on W, this implies that $A_{i_{k-1}} \le 2\beta A_{i_{k+1}}$. So, there must be an index $V \in \{0, \ldots, i_{k-1}\}$ such that conditions 2 and 3 of Case (B) with $m = i_{k+1}$ are satisfied. Further, Condition 5 of Case (B) with that V and with $m = i_{k+1}$ is immediate using all the negations of Condition 5 of (C) with $\hat{v} \in \{u + 1, \ldots, i_{k-1}\}$ and with $m = i_k$.

So, if also Condition 4 is satisfied, then the proof is completed; all the conditions of Case (B) with this V and with $m = i_{k+1}$ are satisfied.

Otherwise, consider Case (C) with V, and with $m = i_{k+1}$. Again, conditions 2 and 4 are satisfied from the same reasons as before, and we are left with conditions 3 and 5 of Case (C). Continuing in this process, eventually, all the conditions of Case (B) or all the conditions of Case (C) will be satisfied, as the right terms of Condition 4 of (B) and of Conditions 3 and 5 of (C) increase in this process, and there is a restriction on W. Further, Condition 5 of (C) with $m = i_{n(t)}$ is always satisfied (see footnote 4). □

Proof of Theorem 1: (i) $\underline{W \geq \sum_{j \in \mathcal{N}(t)} \alpha_j}$:

Assume that $(x^*(t), y^*(t))$ satisfies (25). Then, clearly, $x^*(t) \in \mathcal{X}(t)$ and $y^*(t) \in \mathcal{Y}(t)$. Now, as $x_i(t)^* > 0$ for each $i \in \mathcal{N}(t)$, each agent's best response to $x_i^*(t)$ is given by (22), that is, $y_i(t)^{opt} = \min\{1, \frac{\alpha_i}{2\beta\alpha_i}\} = \min\{1, \frac{1}{2\beta}\}$. In particular, $y^*(t)$ is a best response to $x^*(t)$.

On the other hand, given $y^*(t)$, $A_i y_i^*(t) = A_i$ or $A_i y_i^*(t) = \frac{A_i}{2\beta}$ for each $i \in \mathcal{N}(t)$. So, according to (14), the inspector's best response to $y^*(t)$ is to allocate to each agent, by their A_i-order, the maximum amount possible. In particular, $x^*(t)$ is a best response to $y^*(t)$. As $y^*(t)$ is a best response to $x^*(t)$, and $x^*(t)$ is a best response to $y^*(t)$, $(x^*(t), y^*(t))$ is a Nash equilibrium.

(ii)$\underline{\sum_{j=i_1}^{i_{k-1}} \alpha_j < W \leq \sum_{j=i_1}^{i_k} \alpha_j \textit{ for } k < n(t) \textit{ or } \sum_{j=i_1}^{i_{k-1}} \alpha_j < W < \sum_{j=i_1}^{i_k} \alpha_j \textit{ for } k = n(t)}$:

(A):

Assume that $\beta < \frac{1}{2}$, and that $(x^*(t), y^*(t))$ satisfies (26). Then, clearly, $x^*(t) \in \mathcal{X}(t)$ and $y^*(t) \in \mathcal{Y}(t)$. Now, as $x_i^*(t) > 0$ for $i \in \{i_1, \ldots, i_k\}$; for $i = i_1, \ldots, i_k$, each agent's best response to $x_i^*(t)$ is given by (22). That is, $y_i(t)^{opt} = \min\{1, \frac{\alpha_i}{2\beta x_i(t)}\} = \min\{1, \frac{1}{2\beta}\} = 1$ (as $\beta < \frac{1}{2}$) for $i = i_1, \ldots, i_{k-1}$, and $y_{i_k}(t)^{opt} = \min\{1, \frac{\alpha_{i_k}}{2\beta x_{i_k}(t)}\} = \min\{1, \frac{\alpha_{i_k}}{2\beta(W - \sum_{j=i_1}^{i_{k-1}} \alpha_j)}\}$ for $i = i_k$. Now, as $W \leq \sum_{j=i_1}^{i_k} \alpha_j < \sum_{j=i_1}^{i_{k-1}} \alpha_j + \frac{\alpha_{i_k}}{2\beta}$, this implies that $\frac{\alpha_{i_k}}{2\beta(W - \sum_{j=i_1}^{i_{k-1}} \alpha_j)} > 1$, so $y_{i_k}(t)^{opt} = 1$. Further, as $x_i^*(t) = 0$ for $i \in \{i_{k+1}, \ldots, i_{n(t)}\}$, for $i = i_{k+1}, \ldots, i_{n(t)}$, each agent's best response to $x_i^*(t)$ is given by (24). That is, $y_i(t)^{opt} = 1$. Hence, $y^*(t)$ is a best response to $x^*(t)$.

On the other hand, given $y^*(t)$, $A_i y_i^*(t) = A_i$ for each $i \in \mathcal{N}(t)$. By (14), the inspector's best response to $y^*(t)$ is to allocate to each agent, by their A_i-order, the maximum amount possible. In particular, $x^*(t)$ is a best response to $y^*(t)$. As $y^*(t)$ is a best response to $x^*(t)$, and $x^*(t)$ is a best response to $y^*(t)$, $(x^*(t), y^*(t))$ is a Nash equilibrium.

(B):

Assume that there exists a $u \in \{i_{k-1}, \ldots, 0\}$ and an $m \in \{i_k, \ldots, i_{n(t)}\}$, such that conditions 1–5 of Case (B) are satisfied, and that $(x^*(t), y^*(t))$ satisfies (27).

We first show that $x^*(t) \in \mathcal{X}(t)$. For $i = u+1, \ldots, m-1$, $[x_i^*(t) = \frac{\alpha_i A_i}{2\beta A_m} \overset{?}{\leq} \alpha_i] \Leftrightarrow [\frac{A_i}{2\beta} \overset{?}{\leq} A_m]$. Now, according to (14) and to Condition 3 of Case (B), $\frac{A_{m-1}}{2\beta} < \cdots < \frac{A_{u+1}}{2\beta} \leq A_m$, so this is satisfied.

Further, for $i = m$, $[x_m^*(t) = W - \sum_{j=i_1}^{m-1} x_j^*(t) \leq \alpha_m] \Leftrightarrow [W \leq \sum_{j=i_1}^{m-1} x_j^*(t) + \alpha_m] \Leftrightarrow [W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m-1} \alpha_j A_j}{2\beta A_m} + \alpha_m]$. Now, as $W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_m}$ by Condition 4 and as $\frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_m} < \frac{\sum_{j=u+1}^{m-1} \alpha_j A_j}{2\beta A_m} + \alpha_m$ because $\frac{\alpha_m}{2\beta} < \alpha_m$, this is satisfied.

Also, for $i = m$, $[x_m^*(t) = W - \sum_{j=i_1}^{m-1} x_j^*(t) \geq 0] \Leftrightarrow [W \geq \sum_{j=i_1}^{m-1} x_j^*(t)] \Leftrightarrow [W \geq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m-1} \alpha_j A_j}{2\beta A_m}]$, which is Condition 5 and so it is satisfied. Finally, it is clear that $\sum_{j=i_1}^{m} x_j^*(t) = W$, so $x^*(t) \in \mathcal{X}(t)$.

We next show that $y^*(t) \in \mathcal{Y}(t)$. Clearly, as $\beta > \frac{1}{2}$, by Condition 1, $y_i^*(t) \leq 1$ for $i = i_1, \ldots, u$. Further, by (14), $A_{u+1} > \cdots > A_m$, so $y_i^*(t) \leq 1$ for $i = u+1, \ldots, m$. Finally, $y_i^*(t) = 1$ for $i = m+1, \ldots, i_{n(t)}$. So, $y^*(t) \in \mathcal{Y}(t)$.

Now, as $x_i^*(t) > 0$ for $i \in \{i_1, \ldots, m-1\}$, each agent's best response to $x_i^*(t)$ is given by (22) for $i = i_1, \ldots, m-1$. That is, $y_i(t)^{opt} = \min\{1, \frac{\alpha_i}{2\beta x_i^*(t)}\} = \min\{1, \frac{1}{2\beta}\} = \frac{1}{2\beta}$ for $i = i_1, \ldots, u$, $y_i(t)^{opt} = \min\{1, \frac{\alpha_i}{2\beta \frac{\alpha_i A_i}{2\beta A_{i_k}}}\} = \min\{1, \frac{A_{i_m}}{A_i}\} = \frac{A_{i_m}}{A_i}$ for $i = u+1, \ldots, m-1$. Also, note that $x_m^*(t) \geq 0$. Now, if $x_m^*(t) > 0$, then $y_m(t)^{opt} = \min\{1, \frac{\alpha_i}{2\beta(W - \sum_{j=i_1}^{m-1} x_j^*(t))}\} = \min\{1, \frac{\alpha_i}{2\beta(W - \sum_{j=i_1}^{u} \alpha_j - \frac{\sum_{j=u+1}^{m-1} \alpha_j A_j}{2\beta A_m})}\} = 1$, as $W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_m}$ by Condition 4. If $x_m^*(t) = 0$, then agent m's best response to $x_m^*(t)$ is given by (24), that is, $y_m(t)^{opt} = 1$. Finally, as $x_i^*(t) = 0$ for $i \in \{m+1, \ldots, i_{n(t)}\}$; for $i = m+1, \ldots, i_{n(t)}$, each agent's best response to $x_i^*(t)$ is given by (24), that is, $y_i(t)^{opt} = 1$. Hence, $y^*(t)$ is a best response to $x^*(t)$.

On the other hand, given $y^*(t)$,

$$A_i y_i^*(t) = \begin{cases} \frac{A_i}{2\beta} & \text{if } i = i_1, \ldots, u, \\ A_m & \text{if } i = u+1, \ldots, m, \\ A_i & \text{if } i = m+1, \ldots, i_{n(t)}. \end{cases} \tag{37}$$

By (14) and by Condition 2, $\frac{A_{i_1}}{2\beta} > \cdots > \frac{A_u}{2\beta} \geq A_m > \cdots > A_{i_n(t)}$.

So, a best response of the inspector to $y^*(t)$ is to allocate the maximum possible to each agent $i = i_1, \ldots, u$, and the remainder of its budget to allocate randomly and feasibly among agents $i = u+1, \ldots, m$. In particular, $x^*(t)$ is a best response to $y^*(t)$. As $y^*(t)$ is a best response to $x^*(t)$, and $x^*(t)$ is a best response to $y^*(t)$, $(x^*(t), y^*(t))$ is a Nash equilibrium.
(*C*):

Assume that there exists a $u \in \{i_{k-1}, i_{k-2}, \ldots, i_1, 0\}$ and an $m \in \{i_k, \ldots, i_{n(t)}\}$ such that conditions 1–5 of Case (C) are satisfied, and that $(x^*(t), y^*(t))$ satisfies (28) with $\lambda \equiv \sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta(W - \sum_{j=i_1}^{u} \alpha_j)}$.

We first show that $x^*(t) \in \mathcal{X}(t)$. For $i = i = u+1, \ldots, m$, $[x_i^*(t) = \frac{\alpha_i A_i}{2\beta\lambda} \leq \alpha_i] \Leftrightarrow [\frac{A_i}{2\beta} \leq \lambda]$. By (14), $\frac{A_m}{2\beta} < \cdots < \frac{A_{u+1}}{2\beta}$, so if $[\frac{A_{u+1}}{2\beta} \leq \lambda]$, the latter inequality is satisfied. So, $[\frac{A_{u+1}}{2\beta} \leq \lambda] \Leftrightarrow [\frac{A_{u+1}}{2\beta} \leq \sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta(W - \sum_{j=i_1}^{u} \alpha_j)}] \Leftrightarrow [W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{A_{u+1}}]$, which is Condition 3 of (C), and so it is satisfied. Further, $\sum_{j=i_1}^{m} x_j^*(t) = \sum_{j=i_1}^{u} \alpha_j + \sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta\lambda} = \sum_{j=i_1}^{u} \alpha_j + \sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta(\sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta(W - \sum_{j=i_1}^{u} \alpha_j)})} = W$. So, $x^*(t) \in \mathcal{X}(t)$.

We next show that $y^*(t) \in \mathcal{Y}(t)$. Clearly, as $\beta > \frac{1}{2}$, by Condition 1, $y_i^*(t) \leq 1$ for $i = i_1, \ldots, u$. For $i = u+1, \ldots, m$, $[y_i^*(t) = \frac{\lambda}{A_i} \leq 1] \Leftrightarrow [\lambda \leq A_i]$. By (14), $A_m < \cdots < A_{u+1}$, so if $[\lambda \leq A_m]$, the latter inequality is satisfied. So, $[\lambda \leq A_m] \Leftrightarrow [W \geq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_m}]$, which is Condition 4 of Case (C) and so it is satisfied. Finally, $y_i^*(t) = 1$ for $i = m+1, \ldots, i_{n(t)}$. So, $y^*(t) \in \mathcal{Y}(t)$.

Now, as $x_i^*(t) > 0$ for $i \in \{i_1, \ldots, m\}$, each agent's best response to $x_i^*(t)$ is given by (22). That is, $y_i(t)^{opt} = \min\{1, \frac{\alpha_i}{2\beta x_i^*(t)}\} = \min\{1, \frac{1}{2\beta}\} = \frac{1}{2\beta}$ for $i = i_1, \ldots, u$, and $y_i(t)^{opt} = \min\{1, \frac{\alpha_i}{2\beta \frac{\alpha_i A_i}{2\beta\lambda}}\} = \min\{1, \frac{\lambda}{A_i}\} = \frac{\lambda}{A_i}$ for $i = u+1, \ldots, m$, as $\lambda \leq A_{i_k}$ (see above). Finally, as $x_i^*(t) = 0$ for $i \in \{m+1, \ldots, i_{n(t)}\}$, for $i = m+1, \ldots, i_{n(t)}$, each agent's best response to $x_i^*(t)$ is given by (24). That is, $y_i(t)^{opt} = 1$. Hence, $y^*(t)$ is a best response to $x^*(t)$.

On the other hand, given $y^*(t)$,

$$A_i y_i^*(t) = \begin{cases} \frac{A_i}{2\beta} & \text{if } i = i_1, \ldots, u, \\ \lambda & \text{if } i = u+1, \ldots, m, \\ A_i & \text{if } i = m+1, \ldots, i_{n(t)}. \end{cases} \tag{38}$$

Now, by (14), $\frac{A_{i_1}}{2\beta} > \cdots > \frac{A_u}{2\beta}$. Also, $[\frac{A_u}{2\beta} \geq \lambda] \Leftrightarrow [\frac{A_u}{2\beta} \geq \sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta(W - \sum_{j=i_1}^{u} \alpha_j)}] \Leftrightarrow [W \geq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{k} \alpha_j A_j}{A_u}]$, which is Condition 2 of Case (C) and so it is satisfied.

Further, by (14), $A_{m+1} > \cdots > A_{i_{n(t)}}$, so $[\lambda \geq A_{m+1}(> \cdots > A_{i_{n(t)}})] \Leftrightarrow [\sum_{j=u+1}^{m} \frac{\alpha_j A_j}{2\beta(W - \sum_{j=i_1}^{u} \alpha_j)} \geq A_{m+1}] \Leftrightarrow [W \leq \sum_{j=i_1}^{u} \alpha_j + \frac{\sum_{j=u+1}^{m} \alpha_j A_j}{2\beta A_{m+1}}]$, which is Condition 5 of (C) and so it is satisfied.

So, a best response of the inspector to $y^*(t)$ is to allocate the maximum possible to each agent $i = i_1, \ldots, u$, and the remainder of its budget to allocate randomly and feasibly among agents $i = u+1, \ldots, m$. In particular, $x^*(t)$ is a best response to $y^*(t)$. As $y^*(t)$ is a best response to $x^*(t)$, and $x^*(t)$ is a best response to $y^*(t)$, $(x^*(t), y^*(t))$ is a Nash equilibrium. □

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