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Statistical tests of stochastic process models used in the financial theory of insurance companies

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Abstract

This paper presents a statistical test allowing the analyst to determine if a given time series is statistically incompatible with being modeled as a linear or log-linear process. Since the commonly used models for financial time series of interest to insurance professionals are linear or log-linear, this paper allows the analyst to verify the linearity of the model under investigation, or else points to the necessity of non-linear modeling. We also show how to test for time series Gaussianity using the same type of statistical test statistic. These results are applied to several financial data sets relevant to the financial operations of insurance companies.

Keywords: Statistical test; Stochastic process model; Financial time series

1. Introduction

Linear time series models have been utilized in the financial modeling of insurance processes for many years. Superposition of the 'efficient markets' and 'rational investor' paradigms from finance in conjunction with the standard linear models frequently used for insurance loss distribution modeling make this a natural outgrowth of the convergence of financial and insurance research (c.f., Smith, 1986; Buhlmann, 1987). Such linear type models occur in insurance, for example, in multivariate regression, polynomial regression, and other linear models used in insurance ratemaking and in credibility theory extended to include temporal aspects. Similarly, ARIMA time series models for interest rate structures in finance and insurance (c.f., Panjer and Bellhouse, 1980; Bellhouse and Panjer, 1981), and factor analysis models using time dependent covariates have been used in arbitrage pricing models to justify low dimensional linear techniques, and then multivariate regression techniques based upon the discovered linear structure have been used to estimate the financial values of equity and bond prices. These linear time series models also form the basis of much of the actuarial literature on ruin theory (based upon compound Poisson and Brownian motion linear processes) related to determining the chance of financial insolvency of insurance firms in a collective risk model. In addition, the Brownian motion process

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(which is linear in continuous time) forms the basis for the Black and Scholes (1973) model of contingent claims modeling.

A linear time series is a time series which can be written in the form

$$X(n) = \sum a(m)u(n-m) \tag{1}$$

where $\{u(n)\}$ is purely random (i.e., i.i.d. mean zero variables). When $a(m) = 0$ for $m < 0$ and $m > q$, then

$$X(n) = \sum_{m=0}^{q} a(m)u(m-n) \tag{2}$$

is called a one-sided (or causal) moving average of order q. It is the independence of $u(n)$ variates which makes a linear process special. A particular case of (2) occurs when the $\{u(n)\}$ are normal random variables. In this case the $X(n)$ process is a Gaussian process. Such stochastic processes (and in particular, the Brownian motion process in the continuous time models which can arise in this manner, c.f., Yeh (1973)) have held a central point in many insurance calculations. For example, the Brownian motion process is often used for modeling in collective risk theory, and, as noted previously, the geometric Brownian motion forms the basis of option pricing models such as that of Black and Scholes (1973) which has been extensively used in insurance modeling of financial structure.

The purpose of this paper is: (1) to provide a review of an empirical time series statistical test which allows the analyst to actually empirically verify or refute the linearity and/or Gaussianity of a given time series process to be examined, and (2) to provide an empirical illustration of this technique as applied to time series which are relevant financial models for use in insurance.

2. Empirically testing for linearity of models

Under certain circumstances financial time series can be approximated by a Brownian motion with drift (c.f. Brockett and Witt, 1991). In addition, such time series models are often postulated as models for the underlying structure used in financial relationship (e.g. in certain investigations of the solvency or financial structure of insurance companies). It is therefore of interest to investigate these models vis-à-vis real data. A more general question to be addressed here is whether or not a particular series is statistically compatible with a linear representation such as that given by (1), and if this series is compatible with having been generated by a Gaussian linear series. We note that testing for univariate or multivariate normality at any given time or fixed set of times is not sufficient to imply Gaussianity of the process since the latter implies not only normality at each individual time point, but also multivariate normality at *each finite collection* of time points. Our development draws from Hinich (1982) and Brockett et al. (1988).

We shall assume that $\{X(n)\}$ is a stationary stochastic process, and without loss of generality, assume that $\mathrm{E}[X(n)] = 0$ (this can be accomplished by centering the series prior to statistical analysis). Working in the frequency domain of the series, we define the *spectrum* of $\{X(n)\}$ to be the Fourier transform of the autocovariance function $C_{xx}(n) = \mathrm{E}[X(t+n)X(t)]$, i.e.,

$$S(f) = \sum_{n=0}^{\infty} C_{xx}(n) \exp\{-2\pi \mathrm{ifn}\}.$$

Turning to higher order frequency coupling, we analogously define the *bispectrum* of $\{X(n)\}$ to be the (two-dimensional) Fourier transform of the third-order moment function $C_{xxx}(n, m) = \mathrm{E}[X(t+n)X(t+m)X(t)]$, i.e.,

$$B(f_1, f_2) = \sum_m \sum_n C_{xxx}(n, m) \exp\{-2\pi \mathrm{i} f_1 n - 2\pi \mathrm{i} f_2 m\}.$$

The bispectrum is a spatially periodic function whose values are completely determined (by symmetry properties of the exponential function) by the values in the principal domain $\Omega = \{0 < 2f_1 < 1; f_2 < f_1; 2f_1 + f_2 < 1\}$.

It measures the extent to which different frequency pairs are simultaneously occurring within the series. Suppose now that $\{X(n)\}$ is a linear time series, that is, it can be expressed as in (1). By stationarity, the joint distribution of $(X(\eta_1), \ldots, X(\eta_N))$ depends only on the time differences between the η_k's. Assuming also that moments exist up to order six for the series, this stationarity also implies that the mean $\mu_x = \mathrm{E}[X(n)]$, the covariance C_{xx}, and the third-order cumulant function $C_{xxx} = \mathrm{E}[x(n+r)X(n+s)X(n)]$ exist and are independent of n.

By the results outlined in Brillinger (1965) and Brillinger and Rosenblatt (1967), the bispectrum of a *linear series* $\{X(t)\}$ can be shown to have the form

$$B(f_1, f_2) = \mu_3 A(f_1) A(f_2) A^*(f_1 + f_2) \tag{3}$$

where $\mu_3 = \mathrm{E}[u^3(t)]$, and

$$A(f) = \sum a(n) \exp(-\mathrm{ifn}), \tag{4}$$

with $A^*(f)$ designating the complex conjugate of $A(f)$. Here u is the innovative series in the representation (1) for X. By a similar calculation, the spectrum of a linear process $X(t)$ can be shown to be

$$S_x(f) = \sigma_u^2 \mid A(f) \mid^2 \tag{5}$$

It then follows from (3) and (5) that for a linear process

$$\frac{\mid B(f_1, f_2) \mid^2}{S_x(f_1) S_x(f_2) S_x(f_1 + f_2)} = \frac{\mu_3^2}{\sigma_u^6}, \text{ a constant.} \tag{6}$$

Hinich (1982) derives a statistical test for linearity and Gaussianity of a time series based upon (6). To do this he first constructs an estimate of the bispectrum $\hat{B}(f_1, f_2)$ and an estimate of the spectrum $\hat{S}(f)$. The ratio of these two estimates $\mid \hat{B}(f_1, f_2) \mid^2 / \hat{S}(\mathrm{f}_1) \hat{S}(\mathrm{f}_2) \hat{S}(\mathrm{f}_1 + \mathrm{f}_2)$ is used to estimate the ratio in (6) at different frequency pairs (f_1, f_2) in the principal domain. Intuitively, if these ratios differ significantly over the collection of different frequency pairs, then you can reject the constancy of the ratio and hence linearity of the time series $\{X(n)\}$. Moreover, because $\mu_3 = 0$ for the normal (or any other symmetric distribution), if the estimated ratios differ too much from zero, then you could reject Gaussianity of the series.

A statistical quantification of the above comparison can be achieved using the interquartile range of the derived ratios because the sampling distributions for $\hat{B}$ and $\hat{S}$ are known. Hence, finding the critical values of the test ratio statistic involves determining the distribution of the interquartile range from a known distribution. This can be derived, and the resulting test statistic can be shown to be asymptotically normal. See Hinich (1982) or Brockett et al. (1988) for precise formulae and more detailed discussion. A computer program to perform this analysis is available from D. Patterson. [1]

3. Applications to series of importance in insurance

We shall use the above bispectral based tests to check certain time series for linearity and Gaussianity. Several time series have been chosen for examination due to their importance in various aspects of financial decision making of insurance companies. These series are inflation rate series, nominal and real interest rate series, and the spot and forward (and the difference between spot and forward) foreign exchange rate series. In addition to these raw series, the first difference of these series is also tested for linearity as an auxiliary test of the random walk hypothesis for the series. Such a random walk model is frequently used in efficient market models.

4. Inflation rates

Knowledge of the behavior of interest rates (both real and nominal) and their relationship with inflation rates is very important to insurance researchers and regulators (c.f., Cameron, 1987; Chang et al., 1985). It is instrumental in addressing such issues as planning defined benefit plans which will remain solvent throughout the pensioners' lifetimes (c.f., Clark and McDermed, 1982), the adequate pricing of variable rate indexed insurance products, the pricing of liability risks accounting for inflation (c.f., McCabe and Witt, 1983), or the implementation of asset liability matching schemes useful for hedging risks potentially leading to insolvency (c.f. Lamm-Tennant, 1989).

For this examination of interest rates and inflation we used as data the yields to maturity from one month T-Bills over the period January 1953 to April 1974. Salomon Brother's publication 'Yields and

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Table 1
Inflation rate linearity and Gaussianity tests

	Reject linearity	Z value	Reject normality	Z value
Δ_t	Yes	17.81	Yes	15.23
$\Delta_t - \Delta_{t-1} = \varepsilon_t$ 1953–74	No	2.21	Yes	5.12
$\Delta_t - \Delta_{t-1} = \varepsilon_t$ 1960–81	No	−0.27	No	2.13
$\Delta_t - \Delta_{t-1} = \varepsilon_t$ 1971–81	No	−0.15	No	0.15

Yield Spreads' is the source of these yields. Inflation rates were taken from the Consumer Price Index for the same period. The period 1953–1974 was chosen for two reasons. First, there was a substantial change in the market basket used to compute the Consumer Price Index in 1953, and second, we wished to be able to compare results with those obtained by Fama (1976) who used the period 1951–1971. To present the results of applying the bispectral tests to the analysis of inflation rates, we let

Δ_t = monthly inflation rate continuously compounded,

and

Δ'_t = monthly 'inflation rate discretely compounded.

The results are presented in Table 1. For large sample sizes the test statistics for linearity and Gaussianity are approximately normally distributed and hence the linearity or Gaussianity of a time series can be checked by looking at the test statistics. For example, a Z-value in a one-tailed test of greater than 2.325 will allow us to reject the linearity hypothesis at a 0.01 level of significance.

As can be seen from Table 1, while the monthly inflation rate series is highly nonlinear and non Gaussian, the first difference of the series appears consistent with a linear Gaussian time series (the random walk model is not refuted) over each subperiod.

5. Real and nominal interest rates

Turning next to the examination of real and nominal interest rates as measured by the T-bill rate, we let

R_t = nominal return on 1 month T-bills,

r_t = real return on 1 month T-bills,

Table 2
Real and nominal interest rate series

	Reject linearity	Z value	Reject normality	Z value
R_t	Yes	117.39	Yes	46.16
$R_t - R_{t-1} = \varepsilon_t$	No	1.17	Yes	3.59
$r_t - r_{t-1} = \varepsilon_t$	Yes	4.00	Yes	4.69

the real rate being calculated by adjusting the nominal interest rate for inflation. Table 2 presents the results of the bispectral tests in these situations.

Here again we find that the raw series are statistically incompatible with either a linear or Gaussian series. In this case, however we find that the nominal interest rate series first differences may be compatible with a linear but still non-Gaussian time series (a non-Gaussian random walk for nominal rates), while the real rates show both nonlinearity and non-Gaussianity in the first differences.

6. Spot and forward foreign exchange rate series

Insurance researchers have begun to investigate the foreign exchange considerations associated with international operations (c.f., Schroath, 1990) and international exchange rate risks (c.f., Schroath and Korth, 1987; Brown and Galitz, 1982). Foreign exchange rate risks and methods for hedging against such risks have been discussed in the insurance literature (e.g., Eldor and Kahane, 1983; Cozzonlino and Jacque, 1987) and have been recognized as important in reinsurance operations (c.f., Louberge, 1983) which in turn are known to significantly relate to solvency of the ceding insurer. The relationship between the futures price and the spot price for foreign exchange rates incorporates this risk. There have been many articles in the finance literature which examine the relationship between the spot price and the corresponding forward price in the foreign exchange market, usually using a linear (often regression or autoregressive) formulation to phrase and test relationships vis-à-vis market efficiency. This relationship is important to insurers and reinsurers globally. (Note that although current regulations in the United States forbid property and liability insurers from using forward contracts, this is not true of insurers in other countries).

Here we present the results of applying the preceding statistical tests to the analysis of the forecast errors in both the original price quote form and also in the log price (rate) form. We have chosen for analysis the U.S. dollar to Japanese Yen exchange rates since this is one of the most closely watched and tightly arbitraged currency exchanges. If linearity and/or Gaussianity is to be found in foreign exchange rates, this is a likely place to find it. We examine two time periods, from January 1, 1981 to mid 1982 and from December 12, 1981 to mid 1983. We have used daily quotes for rates taken from *The Wall Street Journal*, using the thirty day forward rate $F(30, t)$ and the corresponding spot rate $S(30+t)$.

Table 3 shows the results of the analysis applied to the individual spot and forward rate series as well as to the spot and forward forecast errors (spot minus the predicting forward). This examination involves looking at the raw series (spot minus forward), and the rates (log spot minus log forward) and both an additive and multiplicative error structure. The forecast errors examine whether the forward differs from the spot by simple linear noise, and are indicative of the efficiency of the forward rate as predictors of the spot rate as postulated by the foreign exchange literature. The table also shows the results of testing the Gaussinanity.

As is readily apparent, the forecast errors are unequivocally incompatible with linear or Gaussian models. This implies that either a non-linear risk premia is involved in the exchange rate process (c.f., Shome et al. (1988) who have estimated such a non-linear risk premia) or the autoregressive and regressive models postulated by numerous researchers are incorrect. Coincidentally, LeBaren (1993) used the Brock-Dechert-Scheinkman (BDS) test and found that the hourly exchange rate series showed little evidence for linear structure.

7. Conclusions and discussions

This paper has presented a statistical test which allows the researcher to test the adequacy of the linear and/or Gaussian assumption frequently made when modeling an observed time series of financial data. This test was then applied to several series of interest to insurance operations, and it was found that some series were compatible with linear Gaussian series, some with linear but non-Gaussian series, and some were found to be incompatible with either linear or Gaussian series (monthly inflation rate series, nominal interest rate and real rate series, for instance). This has implications for the types of stochastic process models which can be justifiably used in insurance research involving financial series in an efficient capital market.

The past decade witnessed marked advance in time series analysis, especially in non-linear time series modeling and in handling various non-linear

Table 3
Results of test for foreign exchange rate time series linearity and Gaussianity: The U.S. dollar versus Japanese yen [a]

	1/2/81–Mid 82		12/31/81–Mid 83	
	Linearity	Gaussianity	Linearity	Gaussianity
Spot price process $S(t)$	232.749	104.3915	17.669	20.736
30 day forward price process $F(30,t)$	350.266	119.5641	25.418	28.0713
Log spot price process $\ln S(t)$	9.979	25.7844	2.7039	6.6853
Log forward process $\ln F(30,t)$	948.82	164.6701	0.7995	3.9083
Forecast error process $S(t+30)-F(30,t)$	186.39	134.2781	461.134	129.0403
Forecast error process $\ln S(t+30)-\ln F(30,t)$	115.14	104.3095	356.60	106.7738
Forecast error process $S(t+30)/F(30,t)$	277.025	217.5779	239.658	204.594
Forecast error process $\ln S(t+30)/F(30,t)$	77.738	271.9302	190.434	270.0571

[a] All numbers should be compared with standard normal table for significance levels. For comparison purpose, a Z-value of 20 corresponds to a p-value of approximately 10^{-48}. A Z-value of 29 corresponds approximately to a p-value of 10^{-240}. The life of the universe is approximately 10^{100} days.

purturbations to a time series such as outliners, level-shifts and variance changes (Tiao and Tsay, 1994). For example, the idea of repeated stochastic substitution, such as the Gibbs Sampler of Geman and Geman (1984) and the data augmentation of Tanner and Wong (1987), make feasible various computation-intensive statistical methods that were thought to be impossible only a few years ago. Although the advance has been accelerated by the development in computing capabilities and methodologies, more efficient implementation techniques are needed to facilitate the non-linear time series methodologies. Some researcher, for instance, are utilizing parallel computing technique to solve non-linear time series problems (Coleman, 1993). Another reason why people do not use non-linear time series models is that commonly employed statistical procedures cannot tell whether or not the linear hypothesis is rejected exclusively by the existence of non-linearities in the data; the methods currently available in the literature cannot give a clear cut answer to the problem of distinguishing non-linearities from non-stationarities. The rejection of linearity may be due to non-linear deterministic chaos or stochasticity. One way of dealing with this difficulty is to simply assume stochasticity of the process, as we did in this paper. A related problem is the endogenous assumption of the process under study. Indeed, many researchers model the stock index process exogenously, using economic indicators and other factors as explanatory variables, for example as in genetic algorithm models (Bauer, 1994).

The third difficulty employing non-linear time series models is the fact that there have been several developed models, which differ widely, because the real processes are often thought to be chaotic and thus it is hard to understand the underlying structure and mechanism. Among commonly suggested stochastic non-linear time series models are the threshold autoregressive model and the model of conditional heteroskedasticity (Engle, 1982). The bilinear time series model, as a natural non-linear extension of the ARIMA models, can also be an approximation of more general non-linear models (Maravall, 1983). Naturally, applying non-linear time series models to such financial processes as interest rate series, inflation rate series will be our future efforts.

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