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A Goal Programming-Discriminant Function Approach to the Estimation of an Empirical Production Function Based on DEA Results

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Abstract

This paper describes a new frontier estimation procedure which relies on results obtained from Data Envelopment Analysis (DEA). The paper reviews some of the earlier works in this area and points out potential difficulties with them. It further suggests ways to validate such developments. A procedure is constructed on the basis of a Goal Programming (GP)-Discriminant Function model developed in stages during the 1980s. A numerical example is used to illustrate the proposed procedure. Then, an extensive simulation in which the GP-based procedure is compared to three regression-based techniques is presented. The simulation results clearly indicate the superiority of the proposed technique over the regression alternatives.

Keywords. Data Envelopment Analysis (DEA), Frontier Estimation, Goal Programming (GP), Discriminant Function

1. Introduction

Many complex systems in various industries are composed of a number of production units operating under similar circumstances and guided by similar objectives. The production activity of each such unit is typically measured and described in terms of some observed inputs consumed and outputs produced by the unit. Often, and especially in service industries, there are no known rules that govern the interaction among the inputs and outputs. On the contrary, it's quite clear that in many cases one cannot find even two units which perform under identical rules of operation. For example, consider a system of schools within a school district or a state. Each school employs similar kinds of inputs (e.g., teachers, classrooms, etc.) to produce similar kinds of outputs (e.g., SAT scores, attendance rates, etc.). However, due to a variety of reasons (e.g., different management styles of the principals, different characteristics of the community, etc.), no two schools can be said to operate in an identical manner.

However, the ability to represent, even in an approximate manner, the input-output relations by means of a frontier function is an important step in utilizing historical data (observed inputs and outputs) for current and future managerial planning. For example, resource allocation for existing and new units can be based on the frontier function. Questions of priorities, relative importance of production factors and goal setting can all be addressed by this function.

In recent years, there has been a proliferation of frontier estimation activities revolving around Data Envelopment Analysis (DEA), a methodology introduced by Charnes, Cooper and Rhodes [1978] to evaluate the relative efficiencies of Decision Making Units (DMUs). The application of DEA to a set of DMUs provides a categorization of the units into efficient and inefficient subsets. DEA yields projections of inefficient DMUs onto an efficient frontier which can be categorized as a deterministic nonstatistical frontier (Førsund et al. [1980]). This frontier is assumed to be a piece-wise linear concave function (see the discussion of its properties in Charnes et al. [1985]). However, DEA has not provided an explicit formula to describe the input-output relations along the frontier. Instead, it provides only an implicit description of the frontier by listing the efficient DMUs on it and measuring the distance from the frontier to each inefficient DMU.

Attempts to estimate DEA-related production frontiers date back to the early 1980s. One of the first recorded efforts is the work by Banker et al. [1981] who assumed a Cobb-Douglas[1] form for the production function and built it into the DEA constraints set. A two stage process was developed to estimate the elasticities for each DMU and each output separately and then to estimate the DMU's efficiency rating. However, the estimation of the elasticities was done separately for each DMU and was driven not by an objective of estimating the overall frontier but rather by the objective of finding the largest possible efficiency rating for that DMU.

Later, it was thought that the estimation of the empirical production function could be achieved by constructing the facet equations for each piece of the frontier (Charnes et al. [1985], and Yu et al. [1993]) and obtaining exact rates of substitution. This approach proved to be of little practical use. The number of facets is typically very large, it is difficult to determine all (or just the nonoverlapping set of) the facets or to find all the adjacent facets for a given facet and it is impossible to determine exact input ranges which correspond to any given facet (see the numerical illustrations in Charnes et al. [1989]). Furthermore, the existence of multiple optimal solutions to many of the DEA optimizations (when facets are not fully dimensional) compounds the problem as the resulting (alternate) optimal rates of substitution may be drastically different even along a given facet (not to mention differences that are observed when crossing into another facet).

Abandoning the course of exact determination of the frontier, a number of studies tried to combine DEA with regression analyses in order to generate an approximate frontier function. Rhodes and Southwick [1986] used Tobit analysis approach (based on eliminating first the inefficient DMUs) to estimate frontier functions for universities. Cooper and Gallegos [1991] used stochastic frontier techniques in the evaluation of Latin American Airlines. In their case, the inefficient DMUs were not removed from the analysis. Rather, they adjusted only the inputs to their efficient levels while keeping the outputs untouched, then applied OLS to estimate the parameters of the frontier function. Sengupta [1989] developed a series of stochastic DEA formulations where, for example, log-linear and quadratic functions were used to estimate the frontier under different assumptions on the distributions which regulate the behavior of the residual error terms. This work was expanded in Sengupta [1990] who explored a set of parametric transformations to generate alternative parametric and nonparametric estimates of production frontiers. Charnes et al. [1991] formulated an enhanced Goal Programming model with combined Least Absolute Deviation (LAD) with additional constraints to evaluate Army recruiting activities. Banker et al. [1991] applied Stochastic DEA formulation to evaluate the productivity of software projects and Banker and Maindiratta [1992] combined (the nonparametric) DEA with the (parametric) technique of maximum likelihood function to achieve estimates of the frontier. These estimates, however, are computed only for existing DMUs and the method does not attempt to provide a continuous function estimate of the frontier.

Unfortunately, serious econometric pitfalls may be associated with some of these combined DEA-regression approaches. As Ray [1992] points out, when all the units are projected onto the frontier and later used in a LSR procedure, the covariance matrix of the disturbances suffers from heteroscedasticity and is singular. When only the inputs are adjusted to efficiency (but not the outputs) we no longer have the independently identically distributed (IID) property. Even when only the efficient DMUs are used in the LSR procedure, the problems associated with losing the IID assumption remain.

In addition, none of the studies mentioned above provided an accepted validation scheme to test the outcome of the frontier estimation. Some studies used numerical examples to illustrate their models but provided no yardstick to compare with the estimated parameters. The purpose of the present paper is twofold. First, we propose two alternative evaluation criteria that can be considered to validate existing or future frontier estimation techniques. Second, we suggest an approach based on a discriminant function model developed originally by Freed and Glover [1981] and combined it with DEA outcomes to estimate the parameters of the approximated frontier function.

2. Evaluation Criteria

As the number of DEA related frontier estimation procedures steadily increases, so does the need to evaluate and compare their contribution. In this section we suggest two evaluation criteria and later on in this paper we implement one of them. Our suggestions do not mean in any way that we have exhausted the evaluation possibilities—merely, they are meant to open a new avenue of research in this direction.

2.1. *Ability to Retrieve a Known Function*

Most of the frontier estimation studies used real data collected from some system for which a true underlying production function was unknown. To overcome this difficulty it is suggested to start with a known production function and simulate a single theoretical output from a random set of observed inputs for each DMU. Then, apply random efficiency factors to the theoretical outputs to obtain observed outputs. When DEA is consequently applied to the observed input-output data it estimates the efficiency of the DMUs without having the knowledge of the original function that was used to generate them. Later, the same procedure can be repeated with multiple outputs. For example, it might be useful to employ a multiplicative function to generate the theoretical outputs and a mixture of a discrete (for the efficient portion) and a half-normal distributions (for the inefficient portion) to generate the efficiency factors.

Such simulated experiments were conducted before, (e.g., by Banker et al. [1993]). However, these studies were aimed at comparing DEA with regression techniques where the comparisons were limited to the measurement of how close were the efficiency estimates generated by the competing techniques for each DMU to the true efficiency (as simulated). No attempt was done to retrieve information on the parameters which were used in the original frontier function.

Therefore, one way for a frontier estimation technique to prove its credibility is to demonstrate its ability to retrieve the elements of the original function. As a minimal requirement, it should be able to retrieve the correct ordinal ranking of the input elasticities. A more demanding requirement would be to test the estimated function with the original inputs and measure the distance between the estimated and theoretical outputs. Further, it should be demonstrated how these results improve as the range of efficiency factors applied to the original output value is reduced.

2.2. *Consistent Evaluation Over Time*

The whole premise of the frontier estimation effort is based on the notion that there exists some underlying production function that governs the production processes by the different DMUs. This function, which in DEA is actually composed of a collection of functions that provide supports to the observed input-output relationships, may include time dependent variables to describe changes in performance due to, e.g., technological improvements. Such evaluations were first carried out via window analysis (Charnes et al. [1985]) where simple estimation of shifts in the production frontier over time were obtained. More recently Färe et al. [1994], [1995] and others used Malmquist indices to measure technological changes. However, these papers focus more on identifying shifts in the location of the entire frontier and identifying DMUs that were consistently efficient over time rather than on providing approximate production function parameters.

In other works (e.g., Arnold et al. [1995]), it is implicitly assumed that this function is stable over time so that once it's parameters are estimated, it can be used for planning and managing future activities. Surprisingly, these latter works did not attempt to repeat the estimation of the frontier function over time to test the stability of the results. The authors of the present paper have tested a number of the published techniques with real data which was collected for a set of DMUs over time. Running the techniques for each time period separately as well as in different time windows, drastic changes in the estimated parameters were recorded. These changes followed no pattern and could not be explained as technological changes. A production function that is highly unstable over time, namely—whose parameters are not even remotely close to those estimated for previous periods (using the same functional form), is not helpful in practice. Hence, when the true production function is unknown, consistency tests are essential for any application that involves real-world data on stable systems. For example, in a school district's evaluation, it will be impossible to accept that in one quarter teachers were three times as important as other staff while in the next quarter other staff was two times as important as teachers.

3. A Goal Programming Discriminant Function Model

This section adapts to the DEA context a Goal Programming model which was developed by Freed and Glover [1981] and improved in Freed and Glover [1986] and Glover [1990] as a general purpose instrument for discriminating among specified groups of observations. The data for the model includes a vector of observed values for each entity (here, a DMU), and its association with one of several groups. In our case we consider only two groups, efficient and inefficient DMUs. To associate each evaluated DMU with either the efficient (E) or inefficient groups we solve the (Pareto Optimality test) Additive model of DEA (Charnes et al. [1985]). That is, for each DMU_k, $k = 1, \ldots, n$, we solve,

$$\text{Max} \sum_r s_{rk} + \sum_i \sigma_{ik} \tag{1}$$

s.t.

$$\sum_j Y_{rj} \cdot \lambda_j - s_{rk} = Y_{rk}, \forall r$$

$$\sum_j X_{ij} \cdot \lambda_j - \sigma_{ik} = X_{ik}, \forall i$$

$$\sum_j \lambda_j = 1$$

$$\lambda_j, s_{rk}\ \sigma_{ik} \geq 0$$

If $s_{rk}\ \sigma_{ik} = 0\ \forall r, i \Rightarrow k \in E$

There are several advantages in using this particular model over alternative DEA models. First, the use of the Pareto-Optimality concept is accepted in economics and other fields. Second, this model does not attempt to determine (and later justify) specific efficiency scores. Rather, it merely tests for efficiency and in that respect it perfectly fits the objective of estimating a discriminant function that will distinguish between the efficient and inefficient groups.

The next step is to apply a variant of the model developed by Freed and Glover to estimate the parameters of the discriminant function. The original model considered A_j, $j = 1, \ldots, n$ observations, grouped into groups G_k, $k = 1, \ldots, g$. A pair-wise (i.e., $g = 2$) variant of the model was formulated as:

$$\text{Min} \sum_j h_j \cdot \alpha_j - \sum_j c_j \cdot \delta_j \tag{2}$$

s.t.

$$A_j \cdot X - \alpha_j + \delta_j = b, \qquad \forall j \in G_1$$

$$A_j \cdot X + \alpha_j - \delta_j = b, \qquad \forall j \in G_2$$

$$\alpha_j, \delta_j \geq 0; \qquad b, X \text{ unconstrained}$$

where X represents a linear predictor (or a weighting scheme), b is a scalar breakpoint between the two groups, α_j are misclassification deviations that are minimized and δ_j are desired deviations which are maximized in the objective function which uses h_j and c_j as weights. To avoid the trivial solution of $X = 0$ one must append some normalization constraint to the model. Freed and Glover [1986] and Glover [1990] suggested various normalizations to ensure the condition $X \neq 0$.

The model selects a separating hyperplane $A \cdot X = b$ in a way that attempts not only to have all the units in one group above it and all the units in the other group under it (by minimizing the α_j deviations), but also to push the units in the respective groups as far away as possible from the hyperplane (by maximizing the δ_j deviations) so as to provide the sharpest possible distinction. The model is illustrated graphically in Figure 1 for a two-dimensional matrix $A = \{a_1, a_2\}$.

In contrast to the linear relations exhibited in (2), the input-output relations in DEA are characterized by a piece-wise (linear) concave empirical function (see Charnes et al. [1985]). These relations, along with the classification of efficient and inefficient observations, are demonstrated in Figure 2.

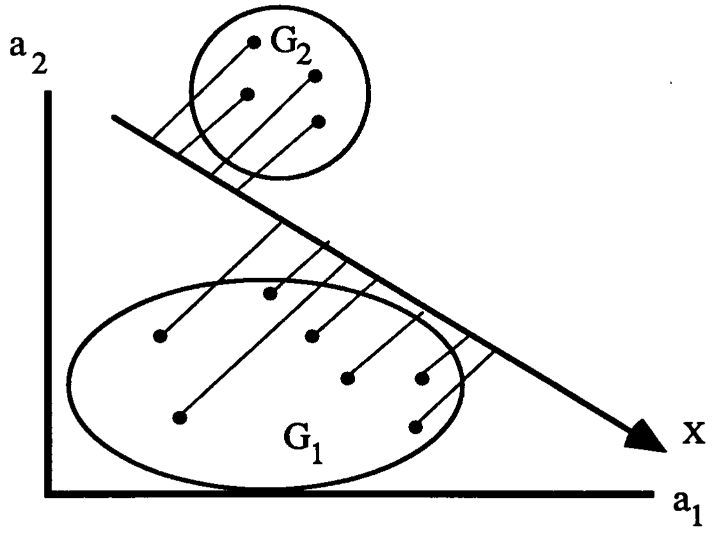

Figure 1. A linear predictor X separating two groups in a 2-dimensional space.

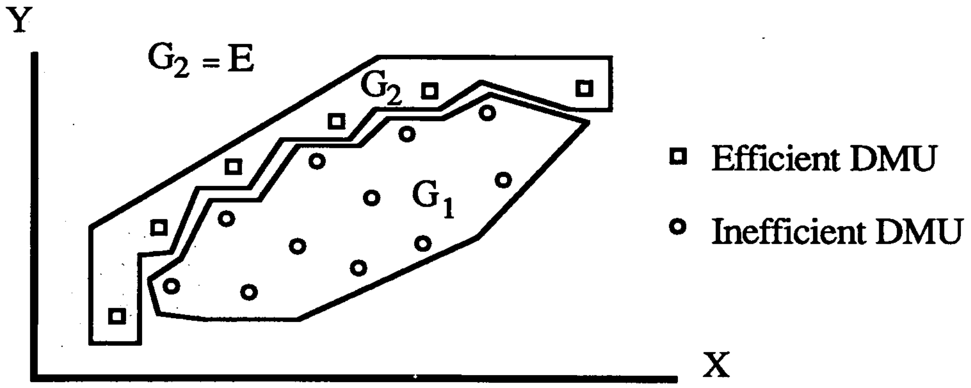

Figure 2. DEA categorization of DMUs into two groups.

Therefore, to adapt the linear model of Freed and Glover to our purposes we must apply some transformation that will convert the concave relations assumed for DEA into a linear function. A common transformation that was used before is a simple logarithmic function (see the introduction). Here, however, we prefer to use a Translog transformation which is more general (it includes the logarithmic one as a private case) and allows for synergy effects among the inputs to be formulated explicitly (see, e.g., Berger [1993] and Banker et al. [1991], footnote 10 on p. 10). Hence, we define:

- the observations $A_j = (\log(Y_{rj}),\ r = 1, \ldots, s;\ -\log(X_{ij}),\ i = 1, \ldots, m;\ -\log(X_{ij}) \cdot \log(X_{kj}),\ i, k = 1, \ldots, m)$
- the groups $G_2 = E$, $G_1 = \neg E$, and
- the linear predictor $X = (\mu, \nu)$.

Thus, our Translog function can be presented as:

$$\sum_r \log(Y_{rj}) \cdot \mu_r = \beta + \sum_i \log(X_{ij}) \cdot \nu_i + \sum_i \sum_k \log(X_{ij}) \cdot \log(X_{kj}) \cdot \nu_{ik}, \quad \forall j \tag{3}$$

Also, following Freed and Glover [1986][2] and without losing generality, we append a normalization on the ν's to prevent an all-zero solution. This leads to the following model:

$$\text{Min} \sum_{j \notin E} h_{1j} \cdot \alpha_j + \sum_{j \in E} h_{2j} \cdot \alpha_j - \sum_{j \notin E} c_{1j} \cdot \delta_j - \sum_{j \in E} c_{2j} \cdot \delta_j \tag{4}$$

s.t.

$$\sum_r \log(Y_{rj}) \cdot \mu_r - \sum_i \log(X_{ij}) \cdot \nu_i - \sum_i \sum_k \log(X_{ij}) \cdot \log(X_{kj}) \cdot \nu_{ik} - \alpha_j + \delta_j = \beta \qquad \forall j \notin E$$

$$\sum_r \log(Y_{rj}) \cdot \mu_r - \sum_i \log(X_{ij}) \cdot \nu_i - \sum_i \sum_k \log(X_{ij}) \cdot \log(X_{kj}) \cdot \nu_{ik} + \alpha_j - \delta_j = \beta \qquad \forall j \in E$$

$$\sum_i \nu_i + \sum_i \sum_k \nu_{ik} = 1$$

$$\alpha_j,\ \delta_j \geq 0,\ \forall j$$

A reader familiar with the DEA literature will find this model similar to the Common Set of Weights formulations developed by Roll et al. [1991], Roll and Golany [1992] and Ganley and Cubbin [1992, p. 89]. In these papers, an optial value of a weight variable (ν or μ) is interpreted as an overall estimate of the relative importance of the particular

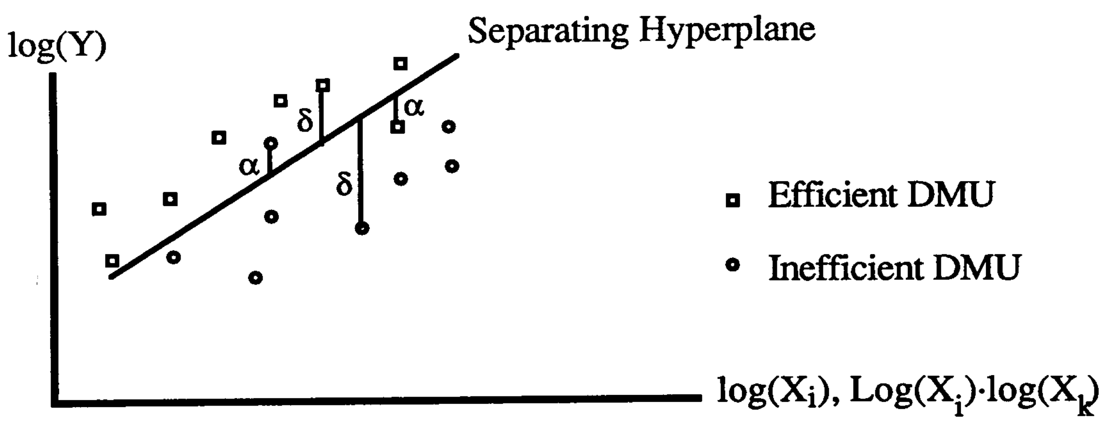

Figure 3. Separating hyperplane with misclassification deviations (α) and distances (δ).

input or output in the production relationship that characterizes the conversion of inputs to outputs across the entire set of DMUs.

The separating hyperplane in the space of the transformed observations as well as the deviations α and δ are shown in Figure 3. Notice that this separation is not *a-priori* assumed to be perfect as we allow some of the efficient units to lie underneath the frontier (a similar approach was first suggested by Timmer [1971]).

4. Solution Procedure

Based on the models presented in the previous section we are now able to propose the following procedure to identify the parameters of a Translog function which can serve as the surrogate to the empirical production function in DEA.

Step I: Solve model (1), associate DMUs with E and $\neg E$.

Step II: Solve the following variant of (4):

$$\text{Min } Z_{\text{II}} \tag{5}$$

s.t.

$$\sum_r \log(Y_{rj}) \cdot \mu_r - \sum_i \log(X_{ij}) \cdot v_i - \sum_i \sum_k \log(X_{ij}) \cdot \log(X_{kj}) \cdot v_{ik} \leq \beta \qquad \forall j \notin E$$

$$\sum_r \log(Y_{rj}) \cdot \mu_r - \sum_i \log(X_{ij}) \cdot v_i - \sum_i \sum_k \log(X_{ij}) \cdot \log(X_{kj}) \cdot v_{ik} + \alpha_j - \delta_j = \beta \qquad \forall j \in E$$

$$\alpha_j \leq Z_{\Pi}, \forall j \in E$$

$$\sum_i v_i + \sum_i \sum_k v_{ik} = 1$$

$$\alpha_j, \delta_j \geq 0, \forall j$$

Drop all DMU in E with $Z_{\Pi} = \alpha_j^*$; repeat this step until $Z_{\Pi} = 0$ or $|E| \leq n'$.

Step III: Solve the following variant of model (4)

$$\text{Min} \sum_{j \in E} (\alpha_j + \delta_j) \tag{6}$$

s.t.

$$\sum_r \log(Y_{rj}) \cdot \mu_r - \sum_i \log(X_{ij}) \cdot v_i - \sum_i \sum_k \log(X_{ij}) \cdot \log(X_{kj}) \cdot v_{ik} \leq \beta \qquad \forall j \notin E$$

$$\sum_r \log(Y_{rj}) \cdot \mu_r - \sum_i \log(X_{ij}) \cdot v_i - \sum_i \sum_k \log(X_{ij}) \cdot \log(X_{kj}) \cdot v_{ik} + \alpha_j - \delta_j = \beta \qquad \forall j \in E$$

$$\sum_i v_i + \sum_i \sum_k v_{ik} = 1$$

$$\alpha_j, \delta_j \geq 0, \forall j$$

The approach outlined in the previous section and summarized in model (4) can now be specified in procedural steps leading to the estimation of the empirical frontier. A key to understanding the procedure is the realization that DEA, being a relative efficiency method, is more likely to misclassify DMUs, which according to an absolute production function are inefficient, as efficient than the opposite (for a statistical analysis of this phenomenon see Korostelev et al. [1992]). If DEA identifies a DMU as inefficient, it means that it has found evidence (i.e., other efficient DMUs) to its inferior position. On the other hand, if DEA identifies a DMU as efficient it only means that it is unable to find such evidence in the observed data, but it does not imply that this DMU is indeed efficient with respect to the unknown production function. Thus, the purpose of Step II, which may be repeated several times, is to drop DMUs in set E from further consideration if they are suspected as being misclassified into E. Thus, in Step II we identify in each iteration the DMU(s) that are associated with the largest misclassification error and drop them from E. This step is repeated until no more misclassification errors are observed for the DMUs in E

or until the number of remaining DMUs in E has reached a minimal level n'. Step III continues to enforce the separation that was reached before (in particular, it allows no misclassification errors for DMUs not in E) while attempting to fit a hyperplane in the center of the reduced efficient set E (see the related work on "Models with Outliers" by Korostelev et al. [1994], pp. 15–20). In that context, Step III can be viewed as a Minimum Absolute Deviation (MAD) regression applied to E with side constraints related to DMUs not in E.

5. A Numerical Example and Simulation

To implement the first validation approach outlined in Section 2, we constructed at random 50 DMUs with four inputs and one output. The input vectors were generated so that the average in each of them was 100, and the theoretical outputs was generated by the multiplicative function:

$$Y = \beta \cdot X_1^{\nu_1} \cdot X_2^{\nu_2} \cdot X_3^{\nu_3} \cdot X_4^{\nu_4} \tag{7}$$

Where the parameters that were chosen were: $\beta = 1$, $\nu_1 = 0.15$, $\nu_2 = 0.4$, $\nu_3 = 0.25$, $\nu_4 = 0.2$. Notice that for the sake of simplicity the sum of the input powers was set to one, i.e., a Constant Returns to Scale (CRS) scenario.

Then, following a procedure used by Banker et al. [1993], the outputs were converted into observed outputs by randomly selecting a third of them to remain at their theoretical values and the rest to be affected by the formula:

$$\hat{Y} = \frac{Y}{1 + \gamma \cdot \text{Rand}} \tag{8}$$

where Rand is a function that generates a random number between zero and one and γ is a scalar. Using $\gamma = 0.5$, formula (8) created inefficiencies (in the form of output underachievements) of up to 1/3 of the original values (the data for this example is provided in the Appendix).

Next, DEA (1) was run on the 50 DMUs using the original inputs and the observed outputs as data. It identified 30 efficient DMUs. Then, the GP procedure given in Section 4 was applied to the same (observed) data with the DEA efficiency categorization. The results obtained by the GP procedure were compared to a Least Squares Regression (LSR) estimation in which the same multiplicative function form was assumed. Similar comparisons between DEA and regression techniques were performed by Bowlin et al. [1985], Banker et al. [1987], and others. The regression was run in four ways: on all the DMUs and only on the efficient DMUs and with and without the constraint that sums all the input powers to one.[3] Table 1 presents the results achieved in this experiment.

Clearly, the GP procedure outperformed its regression based alternatives in this example. However, this outcome may not always be true. To further test the performance of the procedure and compare it with other regression techniques we designed a simulation structure with 11 different settings in which 550 problems were run four times each—resulting in 2200 linear and nonlinear optimizations.[4]

Table 1. Results of the small experiment.

	DMU Set	v_1	v_2	v_3	v_4	Σv_i	Σ (sq. errors)
True Values	All DMUs	0.15	0.4	0.25	0.2	1	—
GP procedure	All DMUs	0.207	0.417	0.186	0.191	1	0.2137
Ordinary LSR	All DMUs	0.226	0.253	0.202	0.311	0.992	0.757
Constrained LSR	All DMUs	0.163	0.232	0.268	0.338	1	0.665
Ordinary LSR	Eff. DMUs	0.304	0.4	0.258	0.368	1.634	0.359
Constrained LSR	Eff. DMUs	0.221	0.304	0.193	0.281	1	0.497

To enable an objective evaluation, the GP and all three regression techniques assumed the same Translog functional form and a normalization summing all input parameters to unity was affected in all programs. In generating the data we used a procedure similar to the one described in (7) and (8) above only this time the functions were expanded from the Cobb-Douglas form used earlier to the Translog functions of the general type shown in (3), that is—each such function had at least one *cross-elasticity* term (right most term in (3)).

Two evaluation criteria were recorded. First, the number of times the GP method outperformed the other techniques and second, the sum of squared (proportional) errors (SSE) for each case. That is:

$$\text{SSE} = \sum_{i=1}^{m} \left(\frac{\hat{v}_i - v_i}{v_i} \right)^2 + \sum_{i,k} \left(\frac{\hat{v}_{ik} - v_{ik}}{v_{ik}} \right)^2 \tag{9}$$

The experimental design for the simulation and its outcomes are given in Table 2.

The design parameters were,

- r: number of outputs (between one and two)
- m: number of inputs (between two and three)
- n: number of DMUs (between 50 and 150)
- p: percent of DMUs left with original outputs (between 20% and 50%)
- γ: extent of randomness in perturbing the outputs—the coefficient that multiplies the random function in the denominator (7)—(between 0.2 and 0.5)
- N: number of problems run (50 for each case)

and the outcome parameters were,

- *M*: number of problems in which the GP method outperformed the other three methods (out of 50)
- *A_GP*: average SSE for the GP procedure
- *A_LSR_E*: average SSE for LSR performed only on efficient DMUs
- *A_LSR_P*: average SSE for LSR performed on all DMUs where inputs of inefficient DMUs were first projected to efficiency
- *A_LSR*: average SSE for LSR performed on all the DMUs

Table 2. Simulation design and outcomes.

r	m	n	p	γ	N	M	A_GP	A_LSR_E	A_LSR_P	A_LSR
1	2	50	20	0.5	50	29	14.55	33.25	59.88	41.82
1	2	50	35	0.5	50	32	11.15	54.85	58.17	92.77
1	2	50	50	0.5	50	40	6.2	25.63	31.06	72.28
1	3	50	35	0.5	50	45	16.18	67.21	109.58	60.60
1	3	100	35	0.5	50	47	13.64	75.47	102.39	88.74
1	3	150	35	0.5	50	48	15.19	141.89	118.66	143.82
2	3	50	35	0.5	50	43	30.81	42.61	65.72	58.38
2	3	100	35	0.5	50	46	20.38	45.25	42.89	62.14
2	3	150	35	0.5	50	45	12.33	202.07	41.91	191.33
2	3	150	35	0.35	50	46	12.45	48.36	25.56	49.77
2	3	150	35	0.2	50	49	12.61	84.99	25.19	90.61

Several important conclusions can be drawn from this simulation:

(i) In all 11 cases studied in this simulation the GP technique has outperformed all three regression-based techniques. In all cases it was the leading method in the majority of the problems solved as well as receiving the best average sum of squared errors.

(ii) From among the regression techniques, only the one that employs all the DMUs in the estimation can be discarded (it was better than the other two only in one out of 11 cases). The other two regression techniques tie in the number of cases in which they lead their group (five each) and therefore one cannot conclude that either one of them is better than the other.

(iii) The first three cases were aimed at testing the effect of leaving a certain proportion of the DMUs with their original outputs. As expected, the larger the proportion becomes the better is the GP estimation (in the extreme case, when $p = 100$, GP will always provide a perfect estimate). No such observation can be made with respect to the regression techniques which experienced changes in both directions in their goodness of fit measures.

(iv) Cases 4–6 were intended to test the effect that the number of DMUs may have on the analysis. The GP results are inconclusive, showing little sensitivity to the number of DMUs. However, the gap between GP and the regression techniques definitely widens with the increase in the number of DMUs. This is due to the fact that the latter use more and more "irrelevant" DMUs in their estimation when more DMUs are present.

(v) Cases 7–9 were run to check the effect that a larger set of input-output variables may have on the results. When compared with cases 4–6 we again find that the results are inconclusive as far as the GP method is concerned. Again, this implies that the GP procedure is not very sensitive to such increases.

(vi) The last three cases (9–11) serve to check the effect that the degree of randomness in perturbing the outputs for inefficient DMUs may have on the results. Unsurprisingly, when the distance between efficient and inefficient DMUs is larger, the GP procedure is doing better since there is a better chance that the preceding DEA model has correctly categorized the majority of the DMUs into their respective groups.

6. Conclusion

This paper addresses a problem that has been extensively researched in the last few years and for which there is no established consensus on the best way to approach it let alone solution procedure that will always perform well. The paper adapts a general purpose goal programming-discriminant function method to the context of estimating empirical production functions resulting from a DEA. As the method belongs to the family of nonstatistical parametric techniques, it shares a limitation which characterizes these kinds of techniques, namely—it does not produce statistical estimates of the standard errors to the estimated parameters.

With this background, it is evidently insufficient to simply present a new technique and demonstrate it on a particular data set. Therefore, we take it two steps further in this paper. First, we focus attention on the validation issues and suggest two alternative approaches to perform it. Then, we follow the first validation approach and present an exhaustive simulation in which we have checked our proposed GP procedure against three regression-based techniques. The simulation clearly indicates the superiority of the new technique over the ones it was compared against. Future research is planned on other flexible function forms as well as an implementation of the second validation approach to real-world data bases.

Notes

1. In fact, it was Farrell [1957], whose work motivated the development of DEA, that first suggested to use Cobb-Douglas functional forms in the estimation of the efficiency frontiers.
2. For a discussion of various normalization schemes and their possible effects on the goal progam-discriminant function see Glover [1990].
3. When $\Sigma v_i \neq 1$, the v's were first normalized to one and then the sum of squared errors computed. The sum of squared errors was computed as in (9).
4. The simulation program was written in C++ with calls to the OSL package of IBM. It was performed on an IBM RS6000 RISC workstation at the University of Texas at Austin. Complete details are available upon request from the authors.

Appendix

DMU	X_1	X_2	X_3	X_4	Y	$\hat{Y}$
1	96.00	68.00	101.00	111.00	87.19	70.78
2	138.85	111.39	189.93	71.34	120.35	83.33
3	117.84	169.76	92.49	80.86	119.94	92.51
4	144.62	141.03	139.50	142.64	141.50	102.48
5	97.16	99.32	77.67	74.50	87.89	72.87
6	76.37	110.32	105.52	69.69	94.18	68.49
7	97.96	109.78	149.48	70.94	106.83	86.20
8	99.76	136.54	82.36	92.44	106.18	83.94
9	85.81	62.99	118.05	73.01	79.51	59.34
10	123.49	84.81	85.30	117.22	95.87	68.46
11	125.78	141.51	52.24	138.06	107.83	79.02

DMU	X_1	X_2	X_3	X_4	Y	$\hat{Y}$
12	125.82	110.99	81.05	127.30	107.46	94.28
13	137.59	142.24	65.20	92.66	106.89	74.32
14	86.89	111.64	115.63	88.43	103.53	83.01
15	56.78	122.42	105.62	131.42	106.64	78.88
16	117.59	143.19	117.48	79.04	117.49	97.00
17	85.94	130.92	81.80	98.64	103.26	81.20
18	127.91	87.62	103.14	113.36	101.70	89.77
19	67.50	106.87	104.13	93.88	96.57	69.90
20	114.65	120.88	56.23	114.70	98.00	90.20
21	88.00	98.00	79.00	90.00	89.83	89.83
22	135.63	134.24	91.92	134.07	122.27	122.27
23	117.38	69.01	141.68	122.69	100.37	100.37
24	119.58	57.55	102.97	122.30	86.36	86.36
25	120.63	82.98	65.45	82.96	82.71	82.71
26	173.79	32.55	162.35	129.65	82.45	82.45
27	38.10	100.59	94.45	90.61	83.83	55.94
28	32.90	65.63	99.18	121.28	74.18	60.81
29	84.76	143.81	27.37	101.77	81.88	81.88
30	87.55	50.72	137.54	68.55	75.02	66.24
31	60.41	100.51	44.50	91.10	74.48	55.31
32	67.66	125.40	64.09	121.52	96.05	80.36
33	108.74	92.56	109.47	137.10	106.97	106.97
34	63.75	89.14	160.88	126.86	105.44	95.60
35	110.97	81.16	36.18	111.77	74.10	74.10
36	209.29	168.09	83.84	79.46	125.67	115.20
37	77.84	80.80	144.26	63.51	88.51	71.22
38	130.88	104.26	85.43	100.63	101.91	101.91
39	121.60	58.13	72.45	126.21	80.12	80.12
40	80.14	99.18	67.91	56.75	78.15	60.40
41	94.26	90.95	70.52	70.97	81.65	77.78
42	127.93	70.81	71.86	77.05	78.99	67.85
43	80.84	61.76	56.65	128.66	72.88	72.88
44	67.68	113.88	149.74	124.07	114.74	107.75
45	116.21	64.22	111.91	68.16	81.62	67.44
46	67.26	95.02	79.30	83.02	83.93	73.29
47	65.03	84.04	146.77	144.97	103.67	97.13
48	66.73	79.37	146.70	67.85	87.38	81.75
49	79.99	107.26	125.85	91.15	103.41	103.41
50	80.14	85.98	146.09	84.00	96.68	94.66

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