Resource Allocation in an Asymmetric Technology Race with Temporary Advantages

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Abstract: We consider two opponents that compete in developing asymmetric technologies where each party's technology is aimed at damaging (or neutralizing) the other's technology. The situation we consider is different than the classical problem of commercial R&D races in two ways: First, while in commercial R&D races the competitors compete over the control of market share, in our case the competition is about the effectiveness of technologies with respect to certain capabilities. Second, in contrast with the "winner-takes-all" assumption that characterizes much of the literature on this field in the commercial world, we assume that the party that wins the race gains a temporary advantage that expires when the other party develops a superior technology. We formulate a variety of models that apply to a one-sided situation, where one of the two parties has to determine how much to invest in developing a technology to counter another technology employed by the other party. The decision problems are expressed as (convex) nonlinear optimization problems. We present an application that provides some operational insights regarding optimal resource allocation. We also consider a two-sided situation and develop a Nash equilibrium solution that sets investment values, so that both parties have no incentive to change their investments. Naval Research Logistics 59: 128–145, 2012

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1. INTRODUCTION

We consider an asymmetric technology race where party A develops one or more technologies aimed at gaining some kind of an operational advantage over party D, while D attempts to develop one or more countermeasures (CMs) that would neutralize, or at least mitigate, the effect of A's technology. We focus on the "tactical" level where the two parties race, one against the other, in developing asymmetric technologies and CMs of tactical scale. Our article assumes that winning the race yields only a "temporary advantage"—if A completes the development of a certain technology and makes it operational before D is ready with an appropriate CM, then A "gains" a certain reward (often measured in terms of the damage caused to D) per each unit of time until the CMs are ready and operational. If D wins the race, that is, it is ready to deploy a CM that is effective against a certain technology that A develops but has not yet deployed, then A gains a smaller reward, which can be as low as zero if D's CM is "perfectly" effective against that technology. Each party has limited resources that can be expended on their respective development efforts. We first focus on "one-sided" scenarios in which A has already made its choices and formulate a sequence of constrained nonlinear optimization models that can be used by D to determine the amount of resources it should allocate toward developing CMs in various settings. Then, we analyze "two-sided" scenarios in which both A and D need to simultaneously allocate their resources and determine the existence of Nash Equilibria solutions for the relevant game theoretical models.

Asymmetrical races with temporary advantages are prevalent in defense or security scenarios. For example, during insurgencies (e.g., Iraq, Afghanistan) insurgents develop and deploy new improvised explosive devices, while government forces develop measures to counter that threat. Other defense-related examples include the race between the developers of advanced antitank missiles and developers of defensive (passive and reactive) suits for armored vehicles to protect them from those missiles and electronic warfare where developers of radars race against developers of radar jamming devices (see, e.g., Ch. 16 in Ref. 2).

In commercial R&D settings, the military term "arms race" is often replaced by the term "time-to-market race" that typically implies a symmetric race where two or more

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firms compete in technology markets by developing products aimed at satisfying the same customers' needs (e.g., Nokia competes with Apple on developing evermore advanced smartphones, GM and Toyota compete on developing better cars, etc.). In such situations, the race is over who would gain a larger market share and not about the effectiveness of the technologies. Asymmetric commercial races of the types modeled in this article are much less common. A typical example of such settings is in computers and information security, where developers of firewalls and other antivirus software compete with developers of destructive software such as viruses and trojans, (see, e.g., Refs. 13 and 20). Another example relates to counterfeit currency, where developers of advanced counterfeit technologies (printers, scanners, etc.) compete against developers of counterfeit detectors [17].

Temporary advantages in commercial settings are quite common in markets associated with rapid clockspeed such as the high-tech industry. Several such environments are described by Fine [10] who argues that commercial advantages are "always" of a temporary nature, and the differences among industrial sectors are mainly due to different technological clockspeeds.

Attempts to mathematically model arms races in the defense (or security) arena date back to the 1930s. Albeit, most of these efforts focused at the "strategic" level and not at the tactical or operational level, as we do in this article. As early as 1935, Richardson [27] developed a simple model of coupled differential equations demonstrating the exponential effect of an arms race, the results of which led him to predict the Second World War. In a more recent paper, Hunter [16] follows Richardson's school of thought by analyzing a three state arms race through developing and solving systems of linear difference equations. His modeling approach is drastically different than ours, as it does not focus on the time it takes to develop new weapons or CMs nor on the relations between expenditures and development times; budgets are not modeled explicitly, and there is no game theoretical aspects in his analysis. Etcheson [9] expands this line of works by including discussions of the strategies that each side may choose to optimize its value function. He criticizes followers of the Richardson tradition for adhering to an atomistic and reductionist perspective that denies the relevance of human values and intentions and offers new theoretical foci and methodological techniques. Still, the methods he offers are based on developing and solving systems of equations rather than the constrained nonlinear optimization and noncooperative game theoretical models that are offered in this article. Finally, as far as we know, there has not been any attempt to develop normative models for arms race at the tactical and operational levels.

Symmetric R&D races in commercial settings that typically apply to controlling market share have been studies quite extensively in both the Operations Research and the Economics literature. Much of this literature (e.g., Harris and Vickers [14], Park [24], Lippman and McCardle [21], and Golany and Rothblum [12]) adopts the "winner-takes-all" approach, that is, the first competitor to finish the development process receives all the benefit (profit), while the other competitors gain nothing (and their investments are lost). In particular, Spector and Zuckerman [30] show that under some conditions the investment rates in winner-takes-all cases should increase monotonically as a function of the project status. Jansen [18] considers a "stealth race" among firms developing similar technologies where each competitor does not even know who else is competing, let alone the progress made by other competitors. Jansen reaches a conclusion that contradicts that of Spector and Zuckerman—that under some assumptions the equilibria investment rates should decrease in time. Ali et al. [1] consider a race similar to the one described in this article. However, unlike our model that considers continuous resource allocation, they only investigate two alternatives: "pioneering" and "incremental" projects. Hopp [15] develops a Markov model of sequential R&D investment where successful firms are assumed to have an advantage in follow-up R&D projects. Doraszelski [8] also develops a model for R&D race where successful firms gain some temporary advantage (in his case, they do so by accumulating knowledge). Unlike Hopp's model which assumes a sequence of stages, Doraszelski builds a continuous time model. Another sequential model was developed by Ofek and Sarvary [23] who study dynamic competition from a marketing perspective where resources such as reputation and advertising play a role alongside R&D investment. The dynamics considered in their study are (discrete) multiperiod; before each period, the competing firms decide how to allocate their resources. Based on the decisions of the firms and the stochastic market reaction, a "leader" and "followers" are determined for that period. They analyze the tendency of the leader and followers to invest more (or less) in R&D projects as a function of their current R&D competence and market reputation. The resource allocation decisions in their article are memoryless; they only depend on the current position of the firm (being a leader or not). In our model, investment decisions have some memory; the choice of CMs to be developed depends on the technologies already operational and their effect.

While the arms-race situation lends itself naturally to a game-theoretic setting, there are instances where this may not be the case. In counter-insurgency conflicts the insurgents (A) typically use simple existing technologies, which are well known to the government (D). In these settings, D typically has an elaborate intelligence network that provides information regarding A's intentions and capabilities. Thus, D can systematically estimate threats by A. On the other hand, the CMs are technologically advanced and developed by D under a shroud of secrecy, and thus, A may have very little knowledge about the CMs' development projects and their development status. For this type of situations, we develop one-sided optimization models for D where the strategy of A is fixed and is probabilistically known to D. We formulate the decision problem of D as a convex optimization problem that can be solved with standard algorithms. We complement the modeling of this arms-race situation with a two-sided game-theoretic model, where both sides consider competitive strategies, and obtain Nash equilibrium solutions. Arguably, it may be difficult to rationalize a Nash equilibrium as a realistic outcome in this tactical arms-race setting—in particular, because information may be incomplete. But, the existence of such equilibrium in this setting can indicate that a tactical arms race may be contained.

Some interesting results and policy implications emerge from the models we develop and their analysis. For example, we address (Section 2.3.1) a specific application in which the challenge faced by D is to choose between developing a "quick and weak" CM that will reduce (but not eliminate) the damages caused by a technology that A has already deployed, and an advanced CM that will neutralize the threat but will take longer to develop. Defining the damage reduction as "effectiveness" and the development rate as "efficiency," we show that when the weak CM is effective, then as it becomes less efficient, it will get more of the resource. This trend is reversed if that CM is not effective. Moreover, the amount of resources a weak CM will be allocated is not necessarily monotone in its efficiency; for low efficiency, the resources allocated may increase, but for high efficiency, the resources allocated may actually decrease. Also, as mentioned above, some other operational insights may be derived from the existence of a Nash equilibrium.

This article is organized as follows. In section 2, we set the stage by specifying assumptions, providing notations, and explaining our model development logic. We analyze various scenarios and prove that the resultant optimization models are convex. We also present a real-world application that provides insights about resource allocation among potential CMs. In Section 3, we study a symmetrical competitive situation and prove the existence of a Nash equilibrium solution. In Section 4, we extend the development to general distributions, and in Section 5, we summarize the article and point out possible future extensions.

2. OPTIMIZATION MODELS

As resources invested are traded-off against damage gained or averted, the two parties are faced with the decision problem of how to invest their resources. The combination of the two decision problems results in a nonzero-sum game. Motivated by the one-sided situation discussed in Section 1, we focus in this section on D's problem and assume that A has selected a specific strategy that is probabilistically known to D.

Let $j = 1, \dots, n$ index the technologies that are developed by A and let $i = 1, \dots, m$ index the CMs that are developed by D. For each j, we denote by τ_j the random time it takes A to develop the jth technology and assume that these random variables have finite expectation. Also, for each i, we denote by $\xi_i(x_i)$ the random time it takes D to develop the ith CM, when the budget allocated for its development is $x_i \geq 0$. Each expectation $E[\xi_i(x_i)]$ is assumed to be finite, and the function $E[\xi_i(\cdot)]$ is assumed to be decreasing and concave in x_i; in particular, we assume that $E[\xi_i(0)] < \infty$ and refer to this condition as the "nondegeneracy assumption." In the degenerate case where $E[\xi_i(0)] = \infty$, the cost associated with 0 expenditure is very large; our analysis can then be carried out by imposing a threshold investment level (that D will implement to avoid high costs).

In this section and the next one, we assume that the development times of the technologies and the CMs have exponential distributions and that these times are jointly independent across technologies and CMs. Specifically, τ_j has exponential distribution with parameter λ_j, $j = 1, \dots, n$, and $\xi_i(x_i)$ has exponential distribution with parameter $\mu_i(x_i)$, $i = 1, \dots, m$ (rate ∞ of an exponential distribution means that the corresponding random variable is set deterministically to 0). We refer to λ_j as the development intensity parameter of technology j, and to $\mu_i(x_i)$ as the development intensity function of CM i. It is assumed throughout that each of the functions $\mu_i(\cdot)$ is increasing, concave, and continuously differentiable; the nondegeneracy assumption asserts that $\mu_i(0) > 0$ for each i. The assumption about exponential distributions is common in related literature (e.g., Ref. 1); yet, it is relaxed in Section 4 where we consider general distributions.

Given a finite budget C, the problem of D is to decide how much to invest in each CM so as to minimize the total cost. Here, the total cost consists of the investment itself and the expected cumulative damage over the time period from the moment a certain technology that A developed becomes operational to the moment an effective CM (i.e., one that neutralizes the threat) becomes available. The damage inflicted by A's technology is assumed to be linear in the duration of the exposure, with damage rates dependent on the effectiveness of the available CMs. Damage across A's technologies is assumed to be additive. Later on in this section, we discuss the damage function in more detail.

The following lemma will be useful for our development.

LEMMA 1: Let τ and $\xi(x)$ be independent exponential random variables with positive parameters λ and $\mu(x)$, respectively. Then

$$E[\xi(x) - \min\{\xi(x), \tau\}] = \frac{1}{\mu(x)} - \frac{1}{\mu(x) + \lambda} = \frac{\lambda}{\mu(x)[\mu(x) + \lambda]}; \quad (1)$$

Table 1. Summary of tractable cases addressed in Section 2.

Section	No. of technologies	No. of CMs	CMs' effectiveness	CMs' development
2.1	1	1	Full/partial	Not applicable
2.2	n	m	Full	Parallel
2.3	n	m	Partial	Parallel
2.4	1	2	Partial	Series

further, if $\mu(\cdot)$ is increasing and concave, then $E[\xi(\cdot) - \min\{\xi(\cdot), \tau\}]$ is decreasing and convex.

PROOF: The first equality in (1) follows from standard properties of the exponential distribution and the second equality is trite. Next, the middle expression of (1) can be expressed as $(g \circ \mu)(x)$, where $\circ$ denotes the composition operation over functions and $g(y) = \frac{1}{y} - \frac{1}{y+\lambda}$ for $y > 0$. As $g(\cdot)$ is convex and decreasing and $\mu(\cdot)$ is concave and increasing, standard arguments show that $(g \circ \mu)(\cdot)$ is convex and decreasing. □

In the following subsections, we model and analyze several scenarios, summarized in Table 1.

2.1. One Technology and One CM

In this subsection, we consider the situation where A is developing a single technology and D is developing a single CM. Let d represents the damage per unit-time when A's technology is operational, while the CM is not yet ready. We assume that when it is ready, the CM provides full protection. This "one-on-one" scenario is presented to illustrate the analysis to follow. For convenience, we drop the indices $i(= 1)$ and $j(= 1)$.

The problem that D faces—how to optimally allocate its resource, given a budget C—is

Program I: $$\min W(x) \equiv x + dE[\xi(x) - \min\{\xi(x), \tau\}] \quad \text{s.t. } 0 \le x \le C. \tag{2}$$

By Lemma 1, the objective function of Program I (for the exponential case) is

$$W(x) = x + d\left[\frac{1}{\mu(x)} - \frac{1}{\mu(x)+\lambda}\right]. \tag{3}$$

Lemma 1 further implies that the function $W(x)$ given by (3) is convex. Consequently, the solution of Program I is given by the next lemma (whose proof is standard and is left to the reader).

LEMMA 2: Program I has an optimal solution. If $W'(0) \ge 0$, then the optimal solution of Program I has $x^* = 0$. If $W'(0) < 0$ and $W'(C) \le 0$ then $x^* = C$. Otherwise, $0 \le x^* \le C$.

The convexity of $W(\cdot)$ implies that Program I can be solved by standard methods such as bisection (e.g., Ref. 4, Chapter 8).

The analysis above also applies to situations where the CM only provides a partial protection against A's technology, rather than complete elimination of its potential damage. In such cases, d represents the per unit-time reduction in the damage caused by A's technology due to the availability of the CM, and the benchmark is the damage inflicted by the technology with the CM in place. The objective is then to minimize the sum of the cost of the damage exceeding this benchmark and the cost spent on the development of the CM. For example, if the CM provides (when available) full protection with probability $0 < p < 1$ against damage $\mathbb{D}$ per unit-time of the technology, we will have $d = p\mathbb{D}$.

2.1.1. *Special Case: Operational Technology and Linear Development Intensity Function*

Assume that A's technology is operational, that is, $\lambda = \infty$, and that the development intensity function is linear and is given by $\mu(x) = ax + b$, with $a, b > 0$. Here, a is the rate of the variable development intensity—the contribution of each dollar added to the (temporal) development-budget of the CM—and b is the fixed "no-cost" intensity of obtaining a capability for countering A's technology by other means such as obtaining a CM that is developed by others. In this case,

$$W(x) = x + \frac{d}{ax+b}, \tag{4}$$

the optimal allocation x^* is

$$x^* = \begin{cases} 0 & \text{if } b \ge \sqrt{da} \\ \sqrt{\dfrac{d}{a}} - \dfrac{b}{a} & \text{if } b < \sqrt{da} \le b + aC \\ C & \text{if } b + aC < \sqrt{da}, \end{cases} \tag{5}$$

and the optimal cost is

$$W^* \equiv W(x^*) = \begin{cases} \dfrac{d}{b} & \text{if } b \ge \sqrt{da} \\ 2\sqrt{\dfrac{d}{a}} - \dfrac{b}{a} & \text{if } b < \sqrt{da} \le b + aC \\ C + \dfrac{d}{aC+b} & \text{if } b + aC < \sqrt{da}. \end{cases} \tag{6}$$

Evidently, $x^* = x^*(a, b, C, d)$ is piecewise linear and nonincreasing in b. Moreover, the decision how much to invest (if at all) depends on the difference between two expressions that stem from the CM's characteristics, namely, $\sqrt{da}$ and b.

We next examine the dependence of the optimal cost $W^* = W^*(C)$ on C (with a, b, and d fixed). Let $\gamma \equiv \sqrt{\frac{d}{a}} - \frac{b}{a}$. If

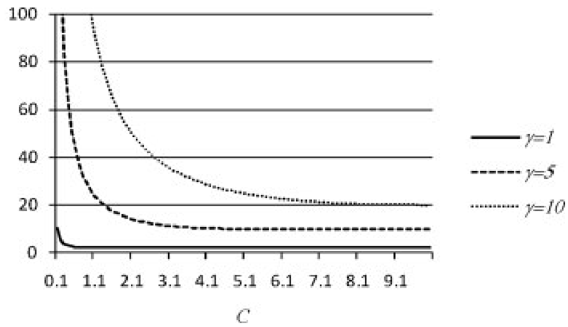

Figure 1. Cost at optimality as a function of the budget for the case $b = 0$.

$b^2 \geq da$, then (6) shows that $W^*(C)$ equals $\frac{d}{b}$, independently of C. Alternatively, if $b^2 < da$, then $W^*(C) = C + \frac{d}{aC+b}$ for $0 \leq C \leq \gamma$ and $W^*(C)$ is constant at $2\sqrt{\frac{d}{a}} - \frac{b}{a}$ for $C > \gamma$. It follows that $W^*(C)$ is convex and nonincreasing (decreasing for $C \in [0, \gamma]$), demonstrating that the budget C exhibits decreasing marginal returns on the optimal cost. Figure 1 depicts $W^*(C)$ for the case where the no-cost capability does not exist, that is, $b = 0$.

2.2. Multiple CMs with 0–1 Effectiveness Developed in Parallel

Suppose m and n are arbitrary and for each technology $j = 1, \ldots, n$ there is a set $I(j) \subseteq \{1, \ldots, m\}$ such that for $i \in I(j)$, CM_i (the ith CM) provides, when available, full protection against technology j. So, technology j is rendered ineffective in the presence of CM_i, $i \in I(j)$. Absent any effective CM (i.e., any $\text{CM}_i i \in I(j)$), technology j causes damage d_j per unit-time. The development of multiple CMs (in parallel) represents a scenario known in the literature as "parallel funding" (see e.g., Refs. 3 and 11). Such situations occur when, due to imminent threat, a defense agency authorizes several contractors to develop in parallel equally effective CMs, although all it really needs is just a single operational CM.

The effect of multiple technologies that A develops is assumed to be additive; the optimization problem that D faces is then the following extension of Program I (with decision vector $x = (x_1, \ldots, x_m)$)

Program II:
$$\min W(x) \equiv \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} d_j E \times \left[\min_{i \in I(j)} \xi_i(x_i) - \min\left\{\min_{i \in I(j)} \xi_i(x_i), \tau_j\right\}\right]$$
$$\text{s.t.} \sum_{i=1}^{m} x_i \leq C, \quad x_i \geq 0 \text{ for } i = 1, \ldots, m. \tag{7}$$

Using Lemma 1, the objective function of Program II (for the exponential case) is

$$W(x) = \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} d_j \left[\frac{1}{\sum_{i \in I(j)} \mu_i(x_i)} - \frac{1}{\sum_{i \in I(j)} \mu_i(x_i) + \lambda_j}\right]. \tag{8}$$

LEMMA 3: The objective function $W(x)$ in (8) is (jointly) convex in $x = (x_1, \ldots, x_n)$.

PROOF: Each of the bracketed terms on the right-hand side of (8) can be expressed as $(g \circ h)(x)$ with $g(y) = d_j\left[\frac{1}{y} - \frac{1}{\lambda+y}\right]$ and $h(x) = \sum_{i \in I(j)} \mu_i(x_i)$; its convexity then follows from standard arguments and the facts that g is convex and decreasing, and h is concave. □

Lemma 3, along with the presence of a polyhedral feasible set, assures that the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for optimality of a feasible solution of Program II. These conditions are next expressed in terms of the n variables $x_1, \ldots, x_n$ that are feasible for Program II, non-negative multipliers $\alpha_1, \ldots, \alpha_n$ corresponding to the non-negativity constraints and a non-negative multiplier β corresponding to the capacity constraint:

$$\alpha_i - \beta = \frac{\partial W(x)}{\partial x_i} = 1 + \mu_i'(x_i) \sum_{j \in J(i)} d_j \times \left[-\frac{1}{\left(\sum_{k \in I(j)} \mu_k(x_k)\right)^2} + \frac{1}{\left(\lambda_j + \sum_{k \in I(j)} \mu_k(x_k)\right)^2}\right], \quad i = 1, \ldots, m, \tag{9}$$

$$x_i \alpha_i = 0, \quad i = 1, \ldots, m \tag{10}$$

and

$$\beta\left(C - \sum_{i=1}^{m} x_i\right) = 0, \tag{11}$$

where for $i = 1, \ldots, m$, $J(i) \equiv \{j = 1, \ldots, n : i \in I(j)\}$. Standard algorithms for convex NLPs can be used to solve Program II, or equivalently, (9)–(11) (e.g., [5, 6, 22]).

2.2.1. *Analysis of Special Cases*

Case 1: Uniform Effectiveness of All CMs with Respect to All Technologies. Here, we assume that $I(j) = \{1, \ldots, n\}$ for each $j = 1, \ldots, n$, i.e., each CM is effective against all of A's technologies. We also assume that the CMs' development intensity functions $\mu_i(x)$ are strictly increasing and strictly concave. For this case, we show a simple method for solving (9)–(11). We first discuss the case where under the optimal solution, say x^*, each CM is allocated a positive amount of the resource, and in addition, the resource is fully utilized. In this case, (10) implies that $\alpha_i = 0$ for each $i = 1, \ldots, n$, and (11) yields no restriction on β (except for non-negativity). By (9), the values of the $\mu_i'(x_i^*)$'s for $i = 1, \ldots, n$ are then equal and their common value, say η, satisfies

$$-1-\beta = \eta \sum_{j=1}^{n} d_j \times \left[-\frac{1}{\left(\sum_{k=1}^{m} \mu_k(x_k^*)\right)^2} + \frac{1}{\left(\lambda_j + \sum_{k=1}^{m} \mu_k(x_k^*)\right)^2} \right]. \quad (12)$$

As each $\mu_i(\cdot)$ is strictly concave, its derivative is strictly decreasing and therefore the derivative has an inverse that we denote $g_i(\cdot)$; in particular, for $i = 1, \ldots, n$, $\mu_i'(x_i^*) = \eta$ is equivalent to $x_i^* = g_i(\eta)$. For each i, let $h_i \equiv \mu_i \circ g_i$, then (12) becomes

$$-1-\beta = \eta \sum_{j=1}^{n} d_j \times \left[-\frac{1}{\left[\sum_{k=1}^{m} h_k(\eta)\right]^2} + \frac{1}{\left[\lambda_j + \sum_{k=1}^{m} h_k(\eta)\right]^2} \right] \quad (13)$$

and the constraint $\sum_{i=1}^{m} x_i = C$ is expressed by $\sum_{i=1}^{m} g_i(\eta) = C$. As each $\mu_i'(\cdot)$ is strictly decreasing, so are the $g_i(\cdot)$'s and $\sum_{i=1}^{m} g_i(\cdot)$. It follows that solving (9)–(11) is reduced to determine η for which the strictly decreasing function $\sum_{i=1}^{m} g_i(\cdot)$ attains the value C. This can be accomplished by bisection (or other methods). If the above calculation leads to $\beta < 0$, then the assumption that the budget is fully utilized in the optimal solution is violated. Further, as $\sum_{i=1}^{m} g_i(\cdot)$ is decreasing, η decreases as the budget C increases (when it is binding); as $x_i^* = g_i(\eta)$ for each $i = 1, \ldots, m$, we conclude that each x_i^* is an increasing function of the budget C.

When some of the optimal x_is are zero, the above approach applies with minor modifications. In this case, the $\mu_i'(x_i)$'s are equal for the indices i with $x_i > 0$, say they equal η. For each z, let $g_i(z) \equiv \max\{(\mu_i')^{-1}(\eta), 0\}$. Solving (9)–(11) is then accomplished by finding η for which the decreasing function $\sum_{i=1}^{m} g_i(\cdot)$ attains the value C, setting $x_i = g_i(\eta)$ for each i and setting the values of α_is so that (9) is satisfied (if $x_i = g_i(\eta) = 0$, then $\eta > (\mu_i')(x_i)$ and α_i determined by (9) is non-negative). It follows that the optimal values x_is are monotone in the budget C.

The above solution method resembles the classic solution of nonlinear (convex) knapsack problems with separable objective functions (e.g., Refs. 4 and 7).

The above analysis extends to situations where each CM only eliminates part of the damage caused by A's technologies, but all CMs have the same effect.

Case 2: Uniform Effectiveness of All CMs with Respect to All Technologies and Linear Development Intensity Functions. Similarly to Section 2.1.1, we assume that the CM development intensity functions are linear and are given by $\mu_i(x) = a_i x + b_i$ with $a_i > 0, b_i \geq 0$, for $i = 1, \ldots, m$. For each i, the parameter a_i is the rate of the variable development intensity, while b_i is the fixed no-cost intensity. If none of A's technologies is operational at time 0, that is, $\lambda_j < \infty$ for each j, the cost function given by (8) becomes

$$\begin{aligned} W(x) &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{m} d_j \left[\frac{1}{\sum_{i=1}^{m} a_i x_i + B} - \frac{1}{\sum_{i=1}^{m} a_i x_i + B + \lambda_j} \right] \\ &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{m} \left[\frac{d_j \lambda_j}{\left(\sum_{i=1}^{m} a_i x_i + B\right)\left(\sum_{i=1}^{m} a_i x_i + B + \lambda_j\right)} \right] \end{aligned}$$

where $B = \sum_{i=1}^{m} b_i$. Using elementary arguments, it can be shown that an optimal solution of Program II is obtained by selecting $i^* \in \arg\max_i a_i$, setting $x_i = 0$ for $i \neq i^*$ and determining x_{i^*} as a maximizer of

$$x + \sum_{j=1}^{m} \left[\frac{d_j \lambda_j}{(a_{i^*} x + B)(a_{i^*} x + B + \lambda_j)} \right] \quad (14)$$

over $0 \leq x \leq C$. If some of A's technologies are operational at time 0, one gets the same conclusions except that the bracketed terms in (14) corresponding operational technologies ($\lambda_j = \infty$) are replaced, respectively, by $\frac{d_j}{a_{i^*} x + B}$. If all of A's technologies are operational at time 0, (14) becomes

$$x + \frac{\Delta}{a_{i^*} x + B}, \quad (15)$$

where $\Delta = \sum_{j=1}^{n} d_j$; the problem of determining the optimal value of x_{i^*} then reduces to the problem considered, solved, and analyzed in Section 2.1.1.

Case 3: One-to-One Technology-CM Correspondence. Here, we assume that $n = m$ and $I(j) = \{j\}$ for each j, that is, there is a one-to-one matching between the set of CMs and the set of A's technologies; each technology can be neutralized by one specific CM and that CM is completely ineffective against the rest of A's technologies. If no technology is operational at time 0, that is, $\lambda_j < \infty$ for each j, the cost function given by (8) becomes

$$W(x) = \sum_{i=1}^{m}\left[x_i + \frac{d_i}{\mu_i(x_i)} - \frac{d_i}{\mu_i(x_i)+\lambda_i}\right] = \sum_{i=1}^{m}\left[x_i + \frac{d_i\lambda_i}{\mu_i(x_i)[\mu_i(x_i)+\lambda_i]}\right].$$

Thus, the objective function $W(x)$ is the sum of m functions, each being a (strictly) convex function of a single decision variable. The problem of minimizing $W(x)$ over $\{x \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^{m} x_i \leq C\}$ is then reduced to an instance of the classic (strictly) convex knapsack problem (e.g., [4]). This reduction also holds when some of A's technologies are operational at time 0; the only difference is that the terms $[x_i + \frac{d_i\lambda_i}{\mu_i(x_i)(\mu_i(x_i)+\lambda_i)}]$ in the above equation corresponding to technologies already operational ($\lambda_j = \infty$) are replaced, respectively, by $[x_i + \frac{d_i}{\mu_i(x_i)}]$. In particular, if all of A's technologies are operational at time 0, we get

$$W(x) = \sum_{i=1}^{m}\left[x_i + \frac{d_i}{\mu_i(x_i)}\right]. \tag{16}$$

Whether or not A's technologies are operational, the cost function $W(x)$ can be expressed as a sum $\sum_{i=1}^{m} f_i(x_i)$, where each f_i is strictly convex. In this case, the KKT conditions (see (9)–(11)) show that the optimal solution x^* is expressed in terms of a multiplier $\beta \geq 0$ such that

$$x_i^* = \begin{cases} 0 & \text{if } f_i'(0) \geq 0 \\ g_i(\beta) & \text{if } f_i'(0) < 0 \end{cases} \tag{17}$$

where g_i is the inverse function of f_i' (as f_i is strictly convex, f_i' is strictly increasing, assuring that g_i is well defined and increasing). Further, with $I \equiv \{i = 1,\ldots,m : f_i'(0) < 0\}$, if $\sum_{i\in I} x_i^* < C$ then $\beta = 0$ and the x_i^*s are independent of C. Alternatively, if $\sum_{i\in I} x_i^* = C$ (i.e., the budget is binding), $\sum_{i\in I} g_i(\beta) = C$, implying that C is increasing in β and therefore for each $i \in I$, $x_i^* = g_i(\beta)$ increases as C increases. Thus, an increase (or decrease) of the budget results in increases (respectively, decreases) in the allocation for each (active) CM.

Case 4: One-to-One Technology-CM Correspondence, Linear Development Intensity Functions and Operational Technologies. Here, the assumption of case 3 is augmented with the assumptions that the CM development intensity functions are linear, expressed by $\mu_i(x) = a_i x + b_i$, with $a_i > 0$ and $b_i \geq 0$ for $i = 1,\ldots,m$, and that all of A's technologies are operational, that is, $\lambda_i = \infty$ for each i. In this case, the above equation reduces to

$$W(x) = \sum_{i=1}^{m}\left[x_i + \frac{d_i}{a_i x_i + b_i}\right]$$

and (9) becomes $\alpha_i - \beta = 1 - \frac{a_i d_i}{(a_i x_i + b_i)^2}$. Thus, if x^* is an optimal solution and $\alpha_1,\ldots,\alpha_n,\beta$ are multipliers satisfying (9)–(10), then for $i = 1,\ldots,n$,

$$\left[x_i^* = 0\right] \Rightarrow \left[0 \leq \alpha_i = 1 + \beta - \frac{a_i d_i}{b_i^2}\right] \Rightarrow \left[(1+\beta)b_i^2 \geq a_i d_i\right],$$

and

$$\left[x_i^* > 0\right] \Rightarrow [\alpha_i = 0] \Rightarrow \left[1 + \beta = \frac{a_i d_i}{\left(a_i x_i^* + b_i\right)^2}\right] \Rightarrow \left[0 < x_i^* = \sqrt{\frac{d_i}{a_i(1+\beta)}} - \frac{b_i}{a_i}\right] \Rightarrow \left[(1+\beta)b_i^2 < a_i d_i\right],$$

implying that

$$x_i^* = \begin{cases} 0 & \text{if } (1+\beta)b_i^2 \geq a_i d_i \\ \sqrt{\dfrac{d_i}{a_i(1+\beta)}} - \dfrac{b_i}{a_i} & \text{if } (1+\beta)b_i^2 < a_i d_i. \end{cases} \tag{18}$$

It follows that each x_i^* is a decreasing function of β, and therefore, it is $\sum_{i=1}^{n} x_i^*$. Hence, when the budget is binding, $C = \sum_{i=1}^{m} x_i^*$ is decreasing in β; on the other hand, when the budget is not binding, $\beta = 0$. We conclude that if the budget is cut from C to $C' < C$, the corresponding multipliers satisfy $\beta' \geq \beta \geq 0$ and the effect on each $x_i^* > 0$ is that the term $-\frac{b_i}{a_i}$ in (20) is preserved, and the term $\sqrt{\frac{d_i}{a_i}}$ is reduced by a fixed factor $\sqrt{\frac{1+\beta}{1+\beta'}} < 1$. The latter reflects the well-known "cut across the board" effect (see Ref. 29).

2.3. Multiple Parallel CMs with Varying Effectiveness Developed in Parallel

Here, we consider situations where D is developing, in parallel, multiple CMs, which can only provide partial protection against A's technologies. Specifically, we assume that the effect of a subset of CMs coincides with that of the most effective CM in that set. We present an explicit optimization problem for determining D's optimal use of its resource and show that this optimization problem is convex and therefore tractable.

We start by considering the case of a single technology ($n = 1$), and rank the m CMs from least to most effective, that is, $d = d^0 \geq d^1 \geq \ldots \geq d^{m-1} \geq d^m$. The parameter d^i expresses the damage rate, when the technology is operational and CM_i is available (possibly with any subset of $\{CM_1,\ldots,CM_{i-1}\}$). Also, d^0 is the damage rate in the absence of any CM. Note the difference between d^i and d_j

used in Section 2.2 (d^i indicates the damage inflicted by a certain technology in the presence of CM_i, whereas d_j represents the damage from technology j in the absence of any CM).

For a given resource allocation $x = (x_1, \ldots, x_m)$ and $i = 0, \ldots, m$, let $\xi^{\langle i \rangle}(x) \equiv \min\{\xi_i(x_i), \ldots, \xi_m(x_m)\}$ and $\widehat{\xi}^{\langle i \rangle}(x) \equiv \min\{\xi^{\langle i \rangle}, \tau\}$, with $\xi^{\langle 0 \rangle}(x) = \widehat{\xi}^{\langle 0 \rangle}(x) = 0$. In particular, $\xi^{\langle i \rangle}(x)$ is the earliest time-instance when CM_i or a more effective CM becomes available.

To simplify notation, we suppress the arguments of $\xi_i(x_i)$, $\xi^{\langle i \rangle}(x)$, and $\widehat{\xi}^{\langle i \rangle}(x)$ and write ξ_i, $\xi^{\langle i \rangle}$, and $\widehat{\xi}^{\langle i \rangle}$, respectively. The time-interval I_i, during which the damage rate is d^i, depends on the value of τ and is given by

$$I_i = \begin{cases} [\xi^{\langle i \rangle}, \xi^{\langle i+1 \rangle}] & \text{if } \tau \le \xi^{\langle i \rangle} \\ (\tau, \xi^{\langle i+1 \rangle}] & \text{if } \xi^{\langle i \rangle} < \tau \le \xi^{\langle i+1 \rangle} \\ \emptyset & \text{if } \tau > \xi^{\langle i+1 \rangle}. \end{cases} \tag{19}$$

The length of I_i in (19) can be expressed by the unified term $[(\xi^{\langle i+1 \rangle} - \xi^{\langle i \rangle}) - (\widehat{\xi}^{\langle i+1 \rangle} - \widehat{\xi}^{\langle i \rangle})]$ and thus the expected damage inflicted by A's technology is expressed by

$$\sum_{i=0}^{m-1} d^i[(\xi^{\langle i+1 \rangle} - \widehat{\xi}^{\langle i+1 \rangle}) - (\xi^{\langle i \rangle} - \widehat{\xi}^{\langle i \rangle})] = \sum_{i=1}^{m} (d^{i-1} - d^i)[\xi^{\langle i \rangle} - \widehat{\xi}^{\langle i \rangle}].$$

It follows that the optimization problem that D faces is

$$\textbf{Program III}: \quad \min W(x) \equiv \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} [d^{i-1} - d^i] \times E[\xi^{\langle i \rangle}(x) - \widehat{\xi}^{\langle i \rangle}(x)]$$
$$\text{s.t.} \sum_{i=1}^{m} x_i \le C \text{ and } x_1, \ldots, x_m \ge 0. \tag{20}$$

As each $\xi_i(x)$ and $\widehat{\xi}^{\langle i \rangle}(x)$ is a minimum of independent exponential random variables, it follows that the objective function of Program III can be expressed by

$$W(x) = \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} [d^{i-1} - d^i] \times \left[\frac{1}{\sum_{k=i}^{m} \mu_k(x_k)} - \frac{1}{\lambda + \sum_{k=i}^{m} \mu_k(x_k)} \right]. \tag{21}$$

LEMMA 4: The objective function $W(x)$ in (21) is jointly convex in $x = (x_1, \ldots, x_m)$.

PROOF: Consider the ith summand in the second term of the left-hand side of (21). It can be expressed as $(g \circ h)(x)$ with $g(y) = [d^{i-1} - d^i]\left[\frac{1}{y} - \frac{1}{\lambda + y}\right]$ and $h(x) = \sum_{k=i}^{m} \mu_i(x_i)$. The convexity of $g \circ h$ follows immediately from the facts that g is convex and decreasing and h is concave. □

Lemma 4, along with the presence of a polyhedral feasible set, assures that the KKT conditions are necessary and sufficient for optimality and that standard convex optimization techniques can be used to solve Program III. But, the solution method described in Section 2.2 does not generalize to the case where the CMs have distinct effects.

We next consider the case where there A developed multiple technologies whose joint damage is additive and the rankings of the effectiveness of the CMs against these technologies are the same. For each i and j, let d_i^j be the per unit-time damage of technology j in the presence of CM_i and let $\widehat{\xi}_j^{\langle i \rangle}(x) \equiv \min\{\xi_j^{\langle i \rangle}(x), \tau_j\}$ with $d = d_j^0 \ge d_j^1 \ge \ldots \ge d_j^{m-1} \ge d_j^m$ for each j. Then, the objective function of Program III (given by (20) and (21)) is

$$\begin{aligned} W(x) &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{n}\sum_{i=1}^{m} \left[d_j^{i-1} - d_j^i\right] E\left[\xi^{\langle i \rangle}(x) - \widehat{\xi}_j^{\langle i \rangle}(x)\right] \\ &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{n}\sum_{i=1}^{m} \left[d_j^{i-1} - d_j^i\right] \\ &\quad \times \left[\frac{1}{\sum_{k=i}^{m} \mu_k(x_k)} - \frac{1}{\lambda_j + \sum_{k=i}^{m} \mu_k(x_k)} \right]. \end{aligned} \tag{22}$$

As the sum of convex functions in x, $W(x)$ is convex in x. We note that the convexity is preserved without the assumption that the effectiveness rankings of the CMs are uniform against all of A's technologies; but, in this case, the second term in (22) becomes more complicated.

2.3.1. Single Operational Technology and Two CMs

For many years, the south-western part of Israel has been subject to mortar and makeshift missile attacks from the Gaza Strip. While quite primitive and inaccurate, these weapons have disrupted the daily life in that region, causing significant economic, social, and psychological damages. Several potential CMs have been proposed to mitigate the effect of these attacks. These CMs range from high-energy Laser beams [31] through high-velocity high-density projectiles (e.g., the land-based Phalanx system [26]), to Iron Dome missile interceptors [19]. The CMs vary in effectiveness and expected development time. The Israeli government, faced with competing proposals for developing CMs for the short-range missiles' threat, has to decide how to allocate its resources among those proposed CMs such that the expected damage is minimized. A common situation is to decide how to allocate resources between two CMs: one that is slow to develop while being very effective (totally eliminating the threat) and another that can be developed faster but is less effective and provides only partial reduction in damage.

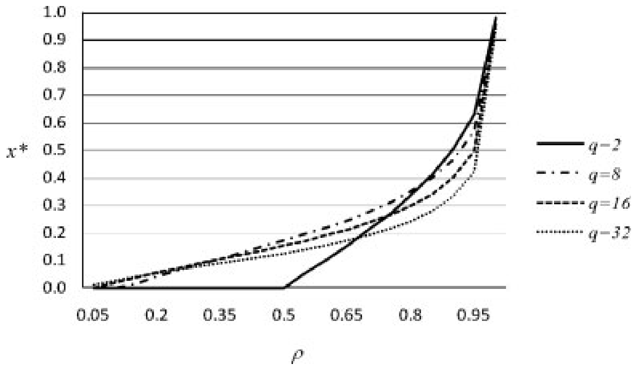

Figure 2. Fraction of resources for CM_1 as a function of the relative effectiveness ρ (parallel development).

Thus, suppose that A's single technology is already operational ($\lambda = \infty$), and D has to decide how to distribute its resources between two possible CMs. While CM_2 can eliminate the threat altogether ($d^2 = 0$), CM_1 is only capable of reducing the damage from its current rate of d^0 per time-period to $0 < d^1 < d^0$. However, with the same amount of resources, CM_1 can be developed faster than CM_2, that is, $\xi_1(x)$ stochastically dominates $\xi_2(x)$ for all x. Without loss of generality, the total budget of D is assumed to be 1. We also assume that the potential damage is very large compared to the resources invested in the development of the CMs and therefore, we exclude the development cost from the objective function. In other words, D will always utilize all of its available resources, and the only question is how to distribute the resources between the two CMs. Let $\mu_i(x_i) = a_i x_i$, where x_i is the fraction of the resource allocated to CM_i, $i = 1, 2$, $x_1 + x_2 = 1$. Let $p \equiv \frac{d^0}{d^1} > 1$, then $\rho = \frac{d^0 - d^1}{d^0} = 1 - p^{-1} < 1$ is the "relative effectiveness" of CM_1. Let $q \equiv \frac{a_1}{a_2}$ be the "development efficiency ratio" of the two CMs. Clearly $q > 1$ must hold, because otherwise the decision is trivial: invest all the resources in the more effective CM_2. The problem is:

$$\min \left[\frac{d^0 - d^1}{a_1 x + a_2(1 - x)} + \frac{d^1}{a_2(1 - x)} \right]$$
$$\text{s.t. } 0 \leq x < 1. \qquad (23)$$

where x is the fraction of resources invested in the less effective CM_1. Problem (23) can be equivalently expressed as

$$\min \left[\frac{p - 1}{(q - 1)x + 1} + \frac{1}{1 - x} \right]$$
$$\text{s.t. } 0 \leq x < 1. \qquad (24)$$

The value of x that minimizes (24) is

$$x^* = \frac{\sqrt{(p - 1)(q - 1)} - 1}{q + \sqrt{(p - 1)(q - 1)} - 1}. \qquad (25)$$

Notice that $0 \leq x^* < 1$, as expected. So, one must always allocate some resources for the "perfect" CM_1. The solution of the optimization problem is expressed in terms of the two parameters—the relative effectiveness $\rho = 1 - p^{-1}$ and the development efficiency ratio q. Notice that for a fixed q, $x^* \to 1$ as $\rho \to 1$; if CM_2 becomes as effective as CM_1, D would better invest all of its resources in the more timely CM. For a fixed p, $x^* \to 0$ as $q \to \infty$; if the development efficiency of the imperfect CM_2 can be increased indefinitely, D should allocate to CM_2 just the required minimum amount, which approaches 0. An interesting limiting behavior is observed when $\rho \to 1 (p \to \infty)$, $q \to \infty$ and $\frac{q}{p} \to \upsilon$. In that case $x^* \to \frac{1}{1+\sqrt{\upsilon}}$. That is, if CM_1 is very effective (very small d^1 value), which implies large investment of resources, and very efficient (very large a_1 value), which implies the sufficiency of a small investment, and the two parameters are improved while maintaining a fixed ratio, υ, then the tradeoff between these two trends is captured by $\frac{1}{1+\sqrt{\upsilon}}$. For example, if $p = q \gg 1$, then one half of the resources should be allocated to the "imperfect" CM_1.

Figure 2 presents the effect of the two parameters—the relative effectiveness ρ and the development efficiency ratio q—on the optimal fraction of resources allocated for CM_1. Clearly, for a fixed q, the fraction of resources allocated to CM_1 is nondecreasing in ρ; more effective CMs receive more resources. For example, if CM_1 reduces damage by only 10%, it gets no resources unless its development rate per unit resource (a_1) is at least 16 times larger than a_2. However, the reverse is not true; for a fixed ρ, x is not necessarily monotone in q, as shown in Fig. 3. For example, if $\rho = 0.9$, which means that CM_1 reduces the damage rate by 90%, then the fraction of resources allocated to CM_1 initially increases with q, as expected, but then decreases as this ratio gets larger. If the development process of CM_1 is very efficient, there is no need to invest much in that CM—it would be more effective to divert resources to the more effective CM_2.

The main policy insight from this analysis is captured in the two figures; at certain ranges of the efficiency ratio q,

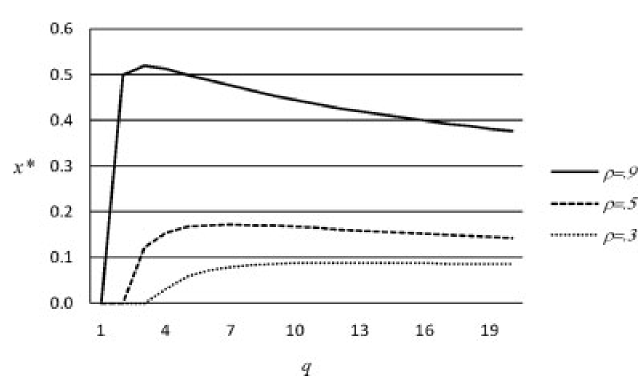

Figure 3. Fraction of resources for CM_1 as a function of the development efficiency ratio q (parallel development).

investment policies may be contradictory for different levels of effectiveness ratios. For example, if CM_1 is quite effective (say $\rho = 0.9$), then the more efficient it is (say, $q \geq 3$), the less resources it will get. However, at the same range, if that CM is relatively ineffective (say $\rho = 0.3$), then the reverse is true; the more efficient CM would receive more resources (up to a certain threshold).

2.4. A Single Technology and Multiple CMs Developed in Series

In this section, we formulate situations with a single technology that was developed by A and multiple CMs developed sequentially, rather than in parallel, by D. We first consider the case of two CMs ($m = 2$), which is shown to be tractable. The first CM, when completed, provides partial response to A's technology. The development of the second CM starts once the first CM is operational. Such situations may occur, when the development of the first CM is an intermediary necessary stage for development of the second CM. Absent any CM, the per unit-time damage of A's technology is d^0. When only the first CM is operational, the damage rate is reduced to $d^1 < d^0$, and when both CMs are operational, the damage rate of A's technology is reduced to 0.

For a given $x = (x_1, x_2)$, let $\bar{\xi}_1$, $\bar{\xi}_2$, and $\bar{\tau}$ be realizations of the development times of the two CMs and A's technology, respectively. If $\bar{\tau} \leq \bar{\xi}_1$, then the damage is $d^0(\bar{\xi}_1 - \bar{\tau}) + d^1(\bar{\xi}_2)$. If $\bar{\xi}_1 < \bar{\tau} \leq \bar{\xi}_1 + \bar{\xi}_2$, then the damage is $d^1(\bar{\xi}_1 + \bar{\xi}_2 - \bar{\tau})$. Finally, if $\bar{\tau} > \bar{\xi}_1 + \bar{\xi}_2$, then there is no damage. For brevity, we shall suppress the dependency on x. The total expected damage is given by

$$P[\tau \leq \xi_1]\{d^0 E[\xi_1 - \tau | \tau \leq \xi_1] + d^1 E[\xi_2]\} + P[\xi_1 < \tau \leq \xi_1 + \xi_2] d^1 E[\xi_1 + \xi_2 - \tau | \xi_1 < \tau \leq \xi_1 + \xi_2]. \quad (26)$$

The next lemma provides explicit representations for the terms in (26).

LEMMA 5: If ξ_1, ξ_2, and τ are independent exponential random variables with parameters μ_1, μ_2, and λ, respectively, then

$$P[\tau \leq \xi_1] = \frac{\lambda}{\lambda + \mu_1}$$

$$P[\xi_1 < \tau \leq \xi_1 + \xi_2] = \frac{\mu_1}{\lambda + \mu_1} \cdot \frac{\lambda}{\lambda + \mu_2}$$

$$E[\xi_1 - \tau | \tau \leq \xi_1] = \frac{1}{\mu_1}$$

$$E[\xi_1 + \xi_2 - \tau | \xi_1 < \tau \leq \xi_1 + \xi_2] = \frac{1}{\mu_2}.$$

PROOF: The first two equalities are standard results about the minimum of independent exponential random variables. The last two equalities follow from the memoryless property of the exponential distribution. □

Lemma 5 allows us to formulate D's decision problem by the following program.

Program IV:

$$\min W(x) \equiv \Bigg\{x_1 + x_2 + \left[\frac{\lambda}{\lambda + \mu_1(x_1)}\right] \times \left[\frac{d^0}{\mu_1(x_1)} + \frac{d^1}{\mu_2(x_2)}\right] + \left[\frac{\mu_1(x_1)}{\lambda + \mu_1(x_1)}\right] \times \left[\frac{\lambda}{\lambda + \mu_2(x_2)}\right]\left[\frac{d^1}{\mu_2(x_2)}\right]\Bigg\}$$

$$\text{s.t. } x_1 + x_2 \leq C \text{ and } x_1, x_2 \geq 0. \quad (27)$$

We observe that the objective function of Program IV can be expressed as

$$\begin{aligned} W(x) &= x_1 + x_2 + d^0\left[\frac{1}{\mu_1(x_1)} - \frac{1}{\lambda + \mu_1(x_1)}\right] \\ &\quad + \left[\frac{\lambda}{\lambda + \mu_1(x_1)}\right]\left[\frac{d^1}{\mu_2(x_2)}\right] \\ &\quad + \left[\frac{\mu_1(x_1)}{\lambda + \mu_1(x_1)}\right]\left[\frac{d^1}{\mu_2(x_2)} - \frac{d^1}{\lambda + \mu_2(x_2)}\right] \\ &= x_1 + x_2 + \left[\frac{d^0}{\mu_1(x_1)} - \frac{d^0}{\lambda + \mu_1(x_1)}\right] + \frac{d^1}{\mu_2(x_2)} \\ &\quad - \left[\frac{\mu_1(x_1)}{\lambda + \mu_1(x_1)}\right]\left[\frac{d^1}{\lambda + \mu_2(x_2)}\right] \\ &= x_1 + x_2 + d^0\left[\frac{1}{\mu_1(x_1)} - \frac{1}{\lambda + \mu_1(x_1)}\right] \\ &\quad + d^1\left[\frac{1}{\mu_2(x_2)} - \frac{1}{\lambda + \mu_2(x_2)}\right] \\ &\quad + d^1\left[\frac{\lambda}{(\lambda + \mu_1(x_1))(\lambda + \mu_2(x_2))}\right]. \end{aligned} \quad (28)$$

LEMMA 6: The objective function $W(x)$ of Program IV is (jointly) convex in $x = (x_1, x_2)$.

PROOF: The first four terms in the sum last expression in (28) are clearly jointly convex in (x_1, x_2). The last term can be expressed as $(g \circ h)(x)$ with $g(y, z) = \frac{1}{yz}$ and $h(x_1, x_2) = \binom{\lambda + \mu_1(x_1)}{\lambda + \mu_2(x_2)}$; now, $g \circ h$ is convex, because g is decreasing and convex (for the latter observe that its Hessian $\begin{pmatrix} 2y^{-3}z^{-1} & y^{-2}z^{-2} \\ y^{-2}z^{-2} & 2y^{-1}z^{-3} \end{pmatrix}$ is positive definite for $y, z > 0$), and h is concave. □

Lemma 6, along with the presence of a polyhedral feasible set, ensures that the KKT conditions are necessary and sufficient for optimality and that standard convex optimization techniques can be used to solve Program IV.

We next outline the extension of the above analysis to multiple CMs that are developed in series. If A's single technology is operational, the damage rate when $CM_1, \ldots, CM_i$ are available is d^i, with $d^1 \geq d^2 \geq \ldots \geq d^m$. Also, as before, in the absence of any CM, the damage rate is $d^0 \geq d^1$.

Consider a given resource allocation $x = (x_1, \ldots, x_m)$ with x_i for $i = 1, \ldots, m$ as the amount of the resource allocated to the development of CM_i. For brevity, we suppress the dependence of μ_i and ξ_i on x_i. Let $T_i \equiv \sum_{u=1}^{i} \xi_u(x_u)$ (with $T_0 = 0$), that is, T_i is the random completion time of CM_i. Extending Lemma 5, we have that for $i = 1, \ldots, m$,

$$P[T_{i-1} < \tau \leq T_i] = \left(\frac{\lambda}{\lambda + \mu_i}\right) \prod_{u=1}^{i-1} \left(\frac{\mu_u}{\lambda + \mu_u}\right), \text{ and}$$

$$E[T_i - \tau | T_{i-1} < \tau \leq T_i] = \frac{1}{\mu_i}.$$

By conditioning on the identity of the interval $(T_{i-1}, T_i]$ in which A's technology becomes operational, we have that the total expected damage is given by

$$\sum_{i=1}^{m} P[T_{i-1} < \tau \leq T_i]\{d^{i-1} E[T_i - \tau | T_{i-1} < \tau \leq T_i] + \sum_{u=i}^{n} d^u E[\xi_{u+1} | T_{i-1} < \tau \leq T_i]\} = \sum_{i=1}^{m} \left(\frac{\lambda}{\lambda + \mu_i}\right) \left[\prod_{u=1}^{i-1} \left(\frac{\mu_u}{\lambda + \mu_u}\right)\right] \left[\sum_{v=i}^{m} \frac{d^{v-1}}{\mu_v}\right].$$

So, the optimization problem that D faces can be expressed by

$$\textbf{Program IV}': \min W(x) \equiv \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} \left(\frac{\lambda}{\lambda + \mu_i}\right) \left[\prod_{u=1}^{i-1} \left(\frac{\mu_u}{\lambda + \mu_u}\right)\right] \left[\sum_{v=i}^{n} \frac{d^{v-1}}{\mu_v}\right]$$
$$\text{s.t. } x_1, \ldots, x_m \geq 0 \text{ and } \sum_{i=1}^{m} x_i \leq C. \quad (29)$$

The objective function of Program IV$'$ with $m = 2$ equals the objective function of Program IV. However, analyzing the convexity of the objective function of Program IV' for $m > 2$ seems to be a difficult task that is hereby posed as an open problem.

2.4.1. *Single Operational Technology, CM_1 Must Be Developed Before CM_2*

Consider a similar situation as in Section 2.3.1. Specifically, $\lambda = \infty, C = 1$ and D has to decide how to distribute its resources between two possible CMs developed in sequence. The less effective CM_1, which is developed first, reduces the damage from its current rate of d^0 per time-period to $0 < d^1 < d^0$. When the perfectly effective CM_2 is completed, then the damage rate of A's technology is reduced to 0. Similar to Section 2.3.1, we assume that the potential damage is very large compared to the resources invested in the development of the CMs, and therefore, the resources are fully utilized. Let $\mu_i(x_i) = a_i x_i$, where x_i is the fraction of the resource allocated to CM_i, $i = 1, 2, x_1 + x_2 = 1$. Let $p \equiv \frac{d^0}{d^1} > 1$, then $\rho = \frac{d^0 - d^1}{d^0} = 1 - p^{-1} < 1$ is the relative effectiveness of CM_1. Let $q \equiv \frac{a_1}{a_2}$ be the development efficiency ratio of the two CMs. The problem is:

$$\min \left[\frac{d^0}{a_1 x} + \frac{d^1}{a_2(1-x)}\right]$$
$$\text{s.t. } 0 \leq x < 1. \quad (30)$$

where x is the fraction of resources invested in the less effective CM_1. Problem (23) can be equivalently expressed as

$$\min \frac{d_1}{a_2} \left[\frac{p}{qx} + \frac{1}{(1-x)}\right]$$
$$\text{s.t. } 0 \leq x < 1. \quad (31)$$

The optimal allocation is then given by

$$x^* = \frac{\sqrt{p}}{\sqrt{p} + \sqrt{q}} \quad (32)$$

and the optimal objective is

$$W^* = W(x^*) = \frac{d_1}{a_2} \left(\sqrt{\frac{p}{q}} + 1\right)^2 = \left(\sqrt{\frac{d_0}{a_1}} + \sqrt{\frac{d_1}{a_2}}\right)^2. \quad (33)$$

Figures 4 and 5 are similar to Figs. 2 and 3, respectively. Figure 6 provides a comparison between the parallel case presented in Section 2.3.1 and the serial case presented here. Notice the profound difference between the parallel and sequential development cases. First, while Fig. 3 is unimodal (see discussion in Section 2.3.1), the comparable plot for the serial case is (strictly) monotone decreasing; the more efficient the development process of the less effective CM that must be developed first, the less resources are needed for its development. Second, as one would expect and as shown in Fig. 6, the parallel case is obviously superior; ceteris paribus,

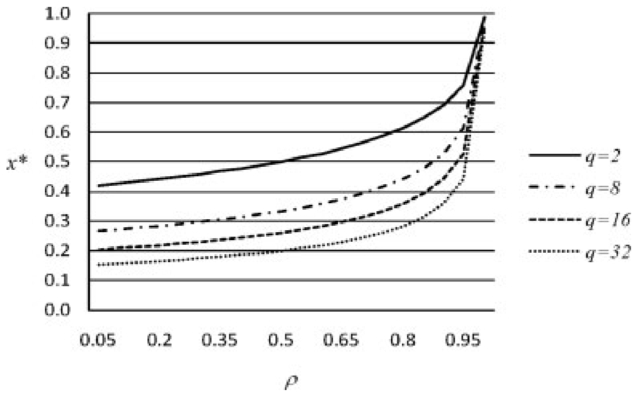

Figure 4. Fraction of resources for CM_1 as a function of the relative effectiveness ρ (serial development).

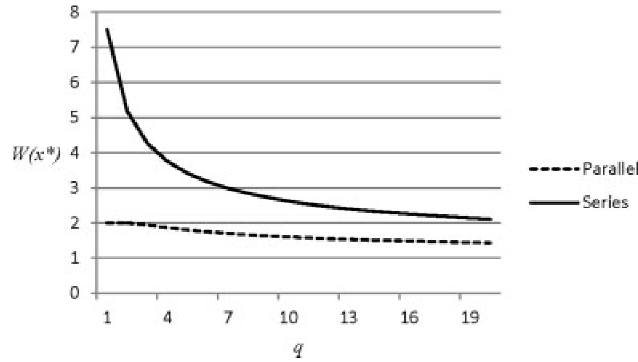

Figure 6. Minimum cost for parallel and serial development, $\rho = 0.5$.

the expected damage from a parallel development situation is always smaller than the damage from a serial development. The gap is significant for small values of q, that is, when the development process of the imperfect CM_1 is inefficient compared to the more effective CM_2. Thus, serial development only occurs when there are manpower or technological constraints such that to reach the ultimate perfect CM, one must pass through an intermediate stage that provides partial effectiveness. Figure 6 illustrates the cost of such constraints, when the damage reduction by the imperfect CM_1 is 50%.

3. EQUILIBRIUM MODEL

In this section, we relax the assumption that the rates at which A develops its technologies are fixed and study a game-theoretic situation where both parties act strategically. Specifically, as for D, the random time it takes A to develop its technologies depends on its resource allocation. The objective of A is to maximize the expected damage inflicted on D less its own development expenses. The objective of D is to minimize the sum of its development expenses and the expected damage that A's technologies will cause.

As in Section 2, the damage inflicted by different technologies is additive, and the damage caused by each technology is proportional to the length of time it is effective. Moreover, the term "damage," is replaced by the term "value of damage" and we let these values to be different for D and A. In other words, the value of the damage inflicted by technology j of A on D is d_j^{D} for D and d_j^{A} for A. For each $j = 1, \ldots, n$, τ_j and λ_j are functions of the amount y_j that A allocates to the development of technology j. The intensities $\lambda_j(\cdot)$ are non-negative increasing, concave, and continuously differentiable. The budget constraint of A is C_{A}, so that the allocation $y = (y_1, \ldots, y_n)$ must satisfy $\sum_{j=1}^{n} y_j \le C_{\mathrm{A}}$. The rest of the assumptions and notation are as in Section 2.

Let $W_{\mathrm{D}}(x, y)$ and $W_{\mathrm{A}}(x, y)$ denote the cost functions of D and A, respectively, and let X and Y denote the set of feasible allocations. We use the superscripts I, II, III, and IV to refer to the four programs of Section 2. A pair $(x^*, y^*) \in X \times Y$ is a Nash equilibrium, if the following two conditions hold

$$W_{\mathrm{D}}(x, y^*) \ge W_{\mathrm{D}}(x^*, y^*) \text{ for each } x \in X, \tag{34}$$

$$W_{\mathrm{A}}(x^*, y) \ge W_{\mathrm{A}}(x^*, y^*) \text{ for each } y \in Y. \tag{35}$$

A classic result of Rosen [28] shows that if the feasible sets X and Y are convex and compact, $W_{\mathrm{D}}(x, y)$ and $W_{\mathrm{A}}(x, y)$ are continuous on $X \times Y$, $W_{\mathrm{D}}(x, y)$ is convex in x for each $y \in Y$, and $W_{\mathrm{A}}(x, y)$ is convex in y for each $x \in X$, then a Nash equilibrium exists.

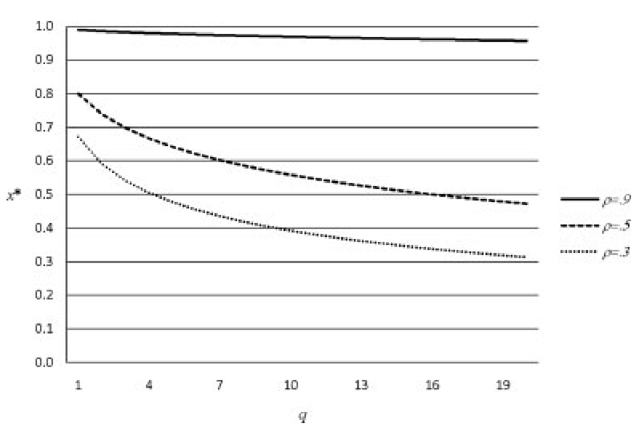

Figure 5. Fraction of resources for CM_1 as a function of the efficiency ratio q (serial development).

3.1. One Technology and One CM

First, we consider the case of a single technology developed by A and single CM developed by D and apply the assumptions of Section 2.1. Allowing λ to be a function of the allocation y of A, the cost functions of D and A are given by:

$$W_{\rm D}^{\rm I}(x,y) = x + d^{\rm D}[E[\xi(x) - \min\{\xi(x),\tau(y)\}] = x + d^{\rm D}\left[\frac{1}{\mu(x)} - \frac{1}{\mu(x)+\lambda(y)}\right], \text{ and} \quad (36)$$

$$W_{\rm A}^{\rm I}(x,y) = y - d^{\rm A}[E[\xi(x) - \min\{\xi(x),\tau(y)\}] = y - d^{\rm A}\left[\frac{1}{\mu(x)} - \frac{1}{\mu(x)+\lambda(y)}\right]. \quad (37)$$

PROPOSITION 1: The two-person game defined by (36) and (37) over $(x, y) \in X \times Y$ has a Nash equilibrium.

PROOF: The arguments of the paragraph following Program I in Section 2 show that $W_{\rm D}(x, y)$ is convex in x. Also, as $\frac{\partial^2 W_{\rm A}}{\partial y^2}(x,y) = -\frac{\lambda''(y)}{[\mu(x)+\lambda(y)]^2} + \frac{2\lambda'^2}{[\mu(x)+\lambda(y)]^3} \geq 0$, $W_{\rm A}(x, y)$ is convex in y. As the feasible sets for both parties are convex and compact, and $W_{\rm A}$ and $W_{\rm D}$ are continuous, the existence of a Nash equilibrium follows from the aforementioned classic result of Rosen [28]. □

3.1.1. *Linear Development Intensity Functions*

Let $\mu(x) = a_{\rm D}x + b_{\rm D}$ and $\lambda(y) = a_{\rm A}y$, where $a_{\rm D} > 0$ and $a_{\rm A} > 0$ are rates of the corresponding variable development intensities and $b_{\rm D} > 0$ is the no-cost capacity of D. Suppressing the superscript "I," we then have

$$W_{\rm D}(x,y) = x + d^{\rm D}\left[\frac{1}{a_{\rm D}x+b_{\rm D}} - \frac{1}{a_{\rm D}x+b_{\rm D}+a_{\rm A}y}\right], \text{ and} \quad (38)$$

$$W_{\rm A}(x,y) = y - d^{\rm A}\left[\frac{1}{a_{\rm D}x+b_{\rm D}} - \frac{1}{a_{\rm D}x+b_{\rm D}+a_{\rm A}y}\right]. \quad (39)$$

We also assume that A and D have enough resources such that there are no effective budget constraints on either sides. This is the case, for example, when $C_{\rm D} > \frac{d^{\rm D}}{b_{\rm D}}$ and $C_{\rm A} > \frac{d^{\rm A}}{b_{\rm D}}$ (for if $x > \frac{d^{\rm D}}{b_{\rm D}}$, then for each $y \geq 0$: $W_{\rm D}(x,y) > x > \frac{d^{\rm D}}{b_{\rm D}} > \frac{d^{\rm D}}{b_{\rm D}} - \frac{d^{\rm D}}{b_{\rm D}+a_{\rm A}y} = W_{\rm D}(0,y)$, and if $y > \frac{d^{\rm A}}{b_{\rm D}}$, then for each $x \geq 0$, $W_{\rm A}(x,y) > y - \frac{d^{\rm A}}{b_{\rm D}} > 0 = W_{\rm A}(x,0)$).

Due to the strict convexity of $W_{\rm D}(x, y)$ in x and of $W_{\rm A}(x, y)$ in y, the KKT conditions are necessary and sufficient; these conditions assert that

$$0 \leq \frac{\partial W_{\rm D}}{\partial x}(x,y) = 1 - \frac{d^{\rm D}a_{\rm D}}{(a_{\rm D}x+b_{\rm D})^2} + \frac{d^{\rm D}a_{\rm D}}{(a_{\rm D}x+b_{\rm D}+a_{\rm A}y)^2}, \quad (40)$$

and

$$0 \leq \frac{\partial W_{\rm A}}{\partial y}(x,y) = 1 - \frac{d^{\rm A}a_{\rm A}}{(a_{\rm D}x+b_{\rm D}+a_{\rm A}y)^2}, \quad (41)$$

with the inequalities in (40) and/or (41) holding as equalities when, respectively, $x > 0$ and/or $y > 0$.

The equality versions of (40) and (41) have a unique solution given by

$$x = \frac{1}{a_{\rm D}}\left[\sqrt{\frac{d^{\rm D}a_{\rm D}}{1+\frac{d^{\rm D}a_{\rm D}}{d^{\rm A}a_{\rm A}}}} - b_{\rm D}\right] = \frac{1}{a_{\rm D}}\left[\sqrt{\frac{1}{\frac{1}{d^{\rm D}a_{\rm D}}+\frac{1}{d^{\rm A}a_{\rm A}}}} - b_{\rm D}\right] \text{ and} \quad (42)$$

$$y = \frac{1}{a_{\rm A}}\left[\sqrt{d^{\rm A}a_{\rm A}} - (b_{\rm D}+a_{\rm D}x)\right] = \frac{1}{a_{\rm A}}\left[\sqrt{d^{\rm A}a_{\rm A}} - \sqrt{\frac{1}{\frac{1}{d^{\rm D}a_{\rm D}}+\frac{1}{d^{\rm A}a_{\rm A}}}}\right]. \quad (43)$$

Let $\Delta \equiv \sqrt{\frac{1}{\frac{1}{d^{\rm D}a_{\rm D}}+\frac{1}{d^{\rm A}a_{\rm A}}}}$. The right-hand side of (43) is always positive, whereas the right-hand side of (42) is positive if and only if $b_{\rm D} < \Delta$. The equality version of (41) has a unique solution (x, y) with $x = 0$—this solution has $y = \frac{\sqrt{d^{\rm A}a_{\rm A}}-b_{\rm D}}{a_{\rm A}}$. Here, $y > 0$ if and only if $b_{\rm D} < \sqrt{d^{\rm A}a_{\rm A}}$ and $(0, y)$ satisfies (40) if and only if $0 \leq 1 - \frac{d^{\rm D}a_{\rm D}}{b_{\rm D}^2} + \frac{d^{\rm D}a_{\rm D}}{d^{\rm A}a_{\rm A}}$, that is,

$$b_{\rm D} \geq \frac{\sqrt{d^{\rm D}a_{\rm D}}}{\sqrt{1+\frac{d^{\rm D}a_{\rm D}}{d^{\rm A}a_{\rm A}}}} = \Delta.$$

The equality version of (40) can never be satisfied by (x, y) with $y = 0$. Finally, (0,0) always satisfies (40), and it satisfies (41) if and only if $b_{\rm D} \geq \sqrt{d^{\rm A}a_{\rm A}}$. It follows that a unique Nash equilibrium exists and is given by

$$(x^*,y^*) = \begin{cases} \left(\frac{\Delta-b_{\rm D}}{a_{\rm D}}, \frac{\sqrt{d^{\rm A}a_{\rm A}}-\Delta}{a_{\rm A}}\right) & \text{if } 0 < b_{\rm D} < \Delta \\ \left(0, \frac{\sqrt{d^{\rm A}a_{\rm A}}-b_{\rm D}}{a_{\rm D}}\right) & \text{if } \Delta \leq b_{\rm D} < \sqrt{d^{\rm A}a_{\rm A}} \\ (0,0) & \text{if } \sqrt{d^{\rm A}a_{\rm A}} \leq b_{\rm D}. \end{cases} \quad (44)$$

The above expression illustrates an intuitive dependence of the (unique) Nash equilibrium on the no-cost capacity of D ($b_{\rm D}$). When $b_{\rm D}$ is small, A and D will invest, respectively, in developing a technology and a CM. When $b_{\rm D}$ is large, neither party will invest. But, when $b_{\rm D}$ is at an intermediary level, A will invest, whereas D will not. A situation where D invests, whereas A does not, will not occur.

It is noted that the above analysis can be extended (with minor modifications) to allow for non-negative no-cost capacity for A.

3.2. Multiple Parallel CMs with 0–1 Effectiveness

Here, we consider the extension of the model discussed in Section 2.2, allowing each λ_j to be a function of y_j. The cost function of D and A are then given by

$$\begin{aligned} W_{\mathrm{D}}^{\mathrm{II}}(x,y) &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} d_j^{\mathrm{D}} E\left[\min_{i\in I(j)} \xi_i(x_i) - \min\left\{\min_{i\in I(j)} \xi_i(x_i), \tau_j\right\}\right] \\ &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} d_j^{\mathrm{D}} \\ &\quad \times \left[\frac{1}{\sum_{i\in I(j)} \mu_i(x_i)} - \frac{1}{\sum_{i\in I(j)} \mu_i(x_i) + \lambda_j(y_j)}\right] \end{aligned} \tag{45}$$

$$\begin{aligned} W_{\mathrm{A}}^{\mathrm{II}}(x,y) &= \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} d_j^{\mathrm{A}} E \\ &\quad \times \left[\min_{i\in I(j)} \xi_i(x_i) - \min\left\{\min_{i\in I(j)} \xi_i(x_i), \tau_j\right\}\right] \\ &= \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} d_j^{\mathrm{A}} \\ &\quad \times \left[\frac{1}{\sum_{i\in I(j)} \mu_i(x_i)} - \frac{1}{\sum_{i\in I(j)} \mu_i(x_i) + \lambda_j(y_j)}\right]. \end{aligned} \tag{46}$$

PROPOSITION 2: The two-person game defined by (45) and (46) has a Nash equilibrium.

PROOF: The conclusion follows from the arguments we used to prove Proposition 1—here the convexity of $W_{\mathrm{D}}^{\mathrm{II}}(x,y)$ in x follows from Lemma 3 and the convexity of $W_{\mathrm{A}}^{\mathrm{II}}(x,y)$ in y follows from the corresponding proof in Proposition 1. □

3.3. Multiple Parallel CMs with Varying Effectiveness

Here, we consider the extension of the model discussed at the end of Section 2.3, allowing each λ_j to be a function of y_j. Of course, random times that were defined in terms of τ_j will now depend on y_j. We explicitly formulate only the case where the rankings of the effectiveness of the CMs against A's technologies are the same (our analysis allows us to relax this assumption but with cumbersome notation). Following (22), the cost functions of D and A are then given by

$$\begin{aligned} W_{\mathrm{D}}^{\mathrm{III}}(x,y) &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{n}\sum_{i=1}^{m} \left[d_j^{i-1,D} - d_j^{i,D}\right] \\ &\quad \times E\left[\xi^{\langle i\rangle}(x) - \widehat{\xi}_j^{\langle i\rangle}(x)\right] \\ &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{n}\sum_{i=1}^{m} \left[d_j^{i-1,D} - d_j^{i,D}\right] \\ &\quad \times \left[\frac{1}{\sum_{k=i}^{m} \mu_k(x_k)} - \frac{1}{\sum_{k=i}^{m} \mu_k(x_k) + \lambda_j(y_j)}\right], \end{aligned} \tag{47}$$

$$\begin{aligned} W_{\mathrm{A}}^{\mathrm{III}}(x,y) &= \sum_{j=1}^{n} y_j - \sum_{j=1}^{n}\sum_{i=1}^{m} \left[d_j^{i-1,A} - d_j^{i,A}\right] \\ &\quad \times E\left[\xi^{\langle i\rangle}(x) - \widehat{\xi}_j^{\langle i\rangle}(x)\right] \\ &= \sum_{j=1}^{n} y_j - \sum_{j=1}^{n}\sum_{i=1}^{m} \left[d_j^{i-1,A} - d_j^{i,A}\right] \\ &\quad \times \left[\frac{1}{\sum_{k=i}^{m} \mu_k(x_k)} - \frac{1}{\sum_{k=i}^{m} \mu_k(x_k) + \lambda_j(y_j)}\right]. \end{aligned} \tag{48}$$

PROPOSITION 3: The two-person game defined by (47) and (48) has a Nash equilibrium.

PROOF: The proof follows from the arguments we used to prove Proposition 1. Here, the convexity of $W_{\mathrm{D}}^{\mathrm{III}}(x,y)$ in x follows from Lemma 4, and the convexity of $W_{\mathrm{A}}^{\mathrm{III}}(x,y)$ in y follows from the corresponding proof in Proposition 1. □

3.4. One Technology and Two CMs Developed in Series

Here, we consider the extension of the model analyzed in Section 2.4 with one technology developed by A and two CMs developed by D ($n = 1$ and $m = 2$), allowing λ to be a function of y. The cost functions of D and A are then given by

$$\begin{aligned} W_{\mathrm{D}}^{\mathrm{IV}}(x,y) &= x_1 + x_2 + d_0^{\mathrm{D}}\left[\frac{1}{\mu_1(x_1)} - \frac{1}{\lambda(y)+\mu_1(x_1)}\right] \\ &\quad + d_1^{\mathrm{D}}\left[\frac{1}{\mu_2(x_2)} - \frac{1}{\lambda(y)+\mu_2(x_2)}\right] \\ &\quad + d_1^{\mathrm{D}}\left[\frac{\lambda(y)}{(\lambda(y)+\mu_1(x_1))(\lambda(y)+\mu_2(x_2))}\right], \end{aligned} \tag{49}$$

$$\begin{aligned} W_{\mathrm{A}}^{\mathrm{IV}}(x,y) &= y - d_0^{\mathrm{A}}\left[\frac{1}{\mu_1(x_1)} - \frac{1}{\lambda(y)+\mu_1(x_1)}\right] \\ &\quad - d_1^{\mathrm{A}}\left[\frac{1}{\mu_2(x_2)} - \frac{1}{\lambda(y)+\mu_2(x_2)}\right] \\ &\quad - d_1^{\mathrm{A}}\left[\frac{\lambda(y)}{(\lambda(y)+\mu_1(x_1))(\lambda(y)+\mu_2(x_2))}\right]. \end{aligned} \tag{50}$$

PROPOSITION 4: The two-person game defined by (49) has a Nash equilibrium.

PROOF: The conclusion follows from the arguments we used to prove Proposition 1. Here, the convexity of $W_{\rm D}^{\rm IV}(x,y)$ in x follows from Lemma 6. Next consider $W_{\rm A}^{\rm IV}(x,y)$. Given $x=(x_1,x_2)$, let $a\equiv\mu_1(x_1)>0$ and $b\equiv\mu_2(x_2)>0$. Then $W_{\rm A}^{\rm IV}(x,y)$ is expressed as a constant plus

$$d_0^{\rm A}\left[\frac{1}{\lambda(y)+a}\right]+d_1^{\rm A}\left[\frac{a}{(\lambda(y)+a)(\lambda(y)+b)}\right].$$

For the convexity of the first term, see the proof of Proposition 1. Next, the convexity of the second term follows from the fact that $\lambda(y)$ is concave in y and that $\frac{1}{(\lambda+a)(\lambda+b)}$ is decreasing and convex in $\lambda\geq 0$; for the latter note that

$$\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{(\lambda+a)(\lambda+b)}\right]=\left[\frac{2}{(\lambda+a)^3(\lambda+b)}+\frac{2}{(\lambda+a)^2(\lambda+b)^2}+\frac{2}{(\lambda+a)(\lambda+b)^3}\right].$$

□

We recall that Section 2.4 includes a formulation of an extension of the model discussed in this subsection to more that two technologies developed by A ($n>2$). We pose the question of determining whether or not the objective functions of D and A are convex in their allocation as an open problem. A positive answer will facilitate the extension of Proposition 4 to $n>2$.

4. GENERAL DISTRIBUTIONS

In this section, we relax the assumption that random completion times of the m technologies and n CMs have exponential distribution and obtain generalizations of Propositions 1–3. Still, we continue to use the notation $\tau_j(y_j)$, $j=1,\dots,m$ and $\xi_i(x_i)$, $i=1,\dots,n$ for the corresponding random variables with y_j and x_i representing, respectively, the allocation of A and D to the development of the jth technology and ith CM. These random variables are assumed to be independent and have finite expectations (we recall that the assumption that the $\tau_j(0)$'s and $\xi_i(0)$'s have finite expectation is referred to as the "nondegeneracy assumption"). But, here, the corresponding cumulative distribution functions (CDFs) of the $\tau_j(y_j)$s and $\xi_i(x_i)$s are denoted $F_j(t,y_j)$ and $G_i(t,x_i)$, respectively. We assume that each of these CDFs has the following properties: for every $t\geq 0$, $F_j(t,y_j)$ and $G_i(t,x_i)$ is increasing, concave and twice continuously differentiable in the decision variable y_j and x_i, respectively, implying that $E[\xi_i(x_i)]=\int_0^\infty(1-G_i(x_i,t))dt$ is decreasing and convex in x_i and, similarly, $E[\tau_j(y_j)]$ is decreasing and convex in y_j.[1] The above assumptions assert that the $E[\xi_i(0)]$s and $E[\tau_j(0)]$s are finite (i.e., nondegeneracy), and they generalize the structure of the exponential case considered in Sections 2 and 3.

LEMMA 7: With x and y representing vector variables, let $\xi(x)$ and $\tau(y)$ be non-negative independent random variables with finite expectation and CDFs $G(t,x)$ and $F(t,y)$, respectively.

(i) If for every $t,x\geq 0$, $F(t,y)$ is increasing and concave in y, then $E[\min\{\xi(x),\tau(y)\}]$ is decreasing and convex in y (for each x).
(ii) If for every $t,y\geq 0$, $G(t,x)$ is increasing and concave in x, then $E[\xi(x)-\min\{\xi(x),\tau(y)\}]$ is decreasing and convex in x (for each y)

PROOF: The CDF of $\min\{\xi(x),\tau(y)\}$ is $1-[1-G(x,t)][1-F(t,y)]$ and therefore (using the tail formula)

$$E[\min\{\xi(x),\tau(y)\}]=\int_0^\infty[1-F(t,y)][1-G(x,t)]dt \tag{51}$$

and

$$\begin{aligned}&E[\xi(x)-\min\{\xi(x),\tau(y)\}]\\&=\int_0^\infty[1-G(x,t)]dt-\int_0^\infty[1-F(t,y)][1-G(x,t)]dt\\&=\int_0^\infty F(t,y)[1-G(x,t)]dt.\end{aligned}\tag{52}$$

For each $t\geq 0$, $1-G(t,x)\geq 0$ and therefore $[1-F(t,y)][1-G(x,t)]$ is decreasing and convex in y. Similarly, for each $t\geq 0$, $F(t,y)\geq 0$ and therefore $F(t,y)[1-G(x,t)]$ is decreasing and convex in x. The asserted conclusions now follow from the fact that the aforementioned monotonicity and convexity are preserved under integration with respect to t. □

Lemma 7 implies that if the cost functions of D and A are given, respectively, by the first expression in (36) and (37) and the CDFs of ξ and τ satisfy the assumptions of Lemma

[1] The assumptions about $G_i(\cdot,\cdot)$ and $F_j(\cdot,\cdot)$ imply that for each "regular" monotonically increasing function $h:[0,\infty)\to\mathbb{R}$, $E[h(\xi_i(x_i))]$ and $E[h(\tau_j(y_j))]$ is decreasing and convex. To see the former, let $g_i(x_i,\cdot)$ be the density function of $\xi_i(x_i)$. Then, $E[h(\xi_i(x_i))]=\int_0^\infty h(t)g_i(x_i,t)dt=-h(\infty)[1-G_i(x_i,\infty)]+h(0)[1-G_i(x_i,0)]+\int_0^\infty h'(t)(1-G_i(x_i,t))dt=\int_0^\infty h'(t)(1-G(x,t))dt$, here regularity means that integration by parts and ignoring the fixed terms is allowed. As $h'(t)\geq 0$ for each t, the conclusions follow.

7, then these cost functions are convex in x and y, respectively. It then follows that the conclusions of Proposition 1 extend to the more general model.

A key tool in the derivation of results of Sections 2 and 3 distribution was the joint convexity of the expectation of the minima of (exponential) development times. The assumptions we have imposed so far are not sufficient of extending this property, and we need an additional property.

LEMMA 8: With $x_1, \ldots, x_m$ representing scalar variables, let $\xi_1(x_1), \ldots, \xi_m(x_m)$ be non-negative, independent random variables with finite expectation and CDFs $G_1(t, x_1), \ldots, G_m(t, x_m)$, respectively. If for every $t \geq 0$, $G_i(t, x_i)$ is increasing and $1 - G_i(t, x_i)$ is log-convex in x_i, then the CDF of $\min_i \xi_i(x_i)$ is increasing and concave in $x = (x_1, \ldots, x_n)$.

PROOF: Our assumptions imply that for every $t \geq 0$, $\log\{\prod_{u=i}^n [1 - G_u(x_u, t)]\} = \sum_{u=i}^n \log[1 - G_u(x_u, t)]$ is (jointly) convex in $x = (x_1, \ldots, x_n)$; hence, $\prod_{u=i}^n [1 - G_u(x_u, t)]$ is log-convex in x and therefore convex. Also, as the $[1 - G_u(x_u, t)]$s are non-negative and decreasing, we have that $\prod_{u=i}^n [1 - G_u(x_u, t)]$ is decreasing in x. The conclusions of the lemma now follow immediately from the observation that the CDF of $\min_i \xi_i(x_i)$ is $1 - \prod_{u=i}^n [1 - G_u(x_u, t)]$. □

We say that the CDFs of the development times of the CMs are "co-log-convex in the resource-allocation" if for each $i = 1, \ldots, m$ ($j = 1, \ldots, n$) and each $t \geq 0$, $\log[1 - G_i(x_i, t)]$ is concave in x_i. Lemma 8 implies that this assumptions assures that the CDFs of the random variables $\min_{i \in I(j)} \xi(x_i)$ appearing in (45) and (46) and the random variables $\xi^{\langle x \rangle}(x)$ appearing in (47) and (48) are increasing and convex in x. Consequently, if the above assumption holds for the $G_i(t, x_i)$s and, in addition, each $F_j(t, y_j)$ (the CDF of the $\tau_j(y_j)$) is increasing and convex in y_j, Lemma7 implies that the cost functions of D and A considered by the first equality of each of the expressions (45) and (46) and (47) and (48) are convex in x and y, respectively. The proofs and conclusions of Propositions 2 and 3 can now be extended to the more general framework. We finally note that the exponential distributions explored in Sections 2 and 3 satisfy the new assumptions.

5. SUMMARY AND EXTENSIONS

This article advances the modeling of R&D races by considering asymmetric situations where winners only sustain temporary advantage with respect to certain operational capabilities. These types of races are mostly prevalent in military conflicts, where weapons developed and deployed on one side are confronted by CMs deployed by the other side. Examples of such situations in the business world relate to criminal activities, such as computer viruses and counterfeiting currency.

The problem of allocating resources in such scenarios can become rather complex, and in this article, we have only started to unveil its layers of complexity. The article develops optimization models (where actions of one party do not affect the decisions of the other party) and then uses the results of these models to construct four propositions that establish the existence of Nash equilibria in scenarios in which actions of each party affect the decisions of the other one.

Section 2.4 considers the case where D is restricted to serial development of CMs: first, an intermediary CM that provides some protection and only later a more advanced CM that provides full protection. Suppose D has the flexibility to choose between serial development of the two CMs versus developing only the advanced CM. For linear intensity functions, the choice between these two options will be determined by the values of the closed form expressions of D's costs as given in (33) and (6) where the former corresponds to developing the two CMs in series and the latter corresponds to developing only the advanced CM.

We note that the Nash equilibrium analysis developed in this article, like many other models based on noncooperative game theoretical concepts that have recently appeared in the operations management literature, assumes that each party has complete information about the parameters that characterizes itself and its rival. Of course, such an assumption can be challenged (because in real-world settings the parties have only partial information about each other). In fact, when we observe the behavior of governments and insurgents, we see that from time to time, one of the parties increases or decreases its investments in developing weapons or CMs. This can be explained as follows. When the race starts, each side has a certain estimate (which might be no more than an educated guess) about the capabilities and intentions of the other party. Both parties solve their respective equations and decide on their investment levels. These decisions hold until one of the parties obtain additional (or more up-to-date) information. Then, new decisions are computed, and a new equilibrium is reached. Clearly, the analysis of such cases is much more complicated, and we leave it to future research.

One possible extension to this article is to allow D to benefit from the CM during the period in which A's technology is not yet operational. For example, during this period, D may deploy its CMs with greater flexibility and then operate them more effectively against A. Still, it is probably safe to assume that the benefit per unit time that D would gain during this period is smaller than the damage per unit time that A would inflict on D, if the CM is not yet available.

Specifically, we can consider a modification of the model where the cost function of D would be

$$\begin{aligned} W_D(x) &= x + d_1 E[\xi(x) - \min\{\tau, \xi(x)\}] \\ &\quad - d_2 E[\tau - \min\{\tau, \xi(x)\}] \\ &= x + d_2 E[\xi(x)] + (d_1 - d_2) E[\xi(x) \\ &\quad - \min\{\tau, \xi(x)\}] - d_2 E[\tau]. \end{aligned} \tag{53}$$

The assumption that the benefit that D gains during the period is secondary to the damage that D suffers when A's technology is developed and the CM is not ready implies that $d_1 > d_2$. As $E[\xi(x)]$ and $E[\xi(x) - \min\{\tau, \xi(x)\}]$ are convex in x, this assumption implies that $W_D(x)$ is convex in x. Consequently, Proposition 1 in Section 3 applies.

Another possible extension relates to the case where the completion time of A's technology depends on the allocation of the resource by A, expressed by the parametric random variable $\tau(y)$. In this case, $W_D(x, y)$ is represented by (53), with $\tau(y)$ replacing τ, in particular, this function is convex in x. Also, the expected cost of A will then be expressed by

$$\begin{aligned} W_A(x, y) &= y - d_3 E[\xi(x) - \min\{\tau(y), \xi(x)\}] \\ &\quad + d_4 E[\tau(y) - \min\{\tau(y), \xi(x)\}]] \\ &= y - d_3 E[\xi(x)] + (d_3 - d_4) E[\min\{\tau(y), \xi(x)\}] \\ &\quad + d_4 E[\tau(y)]. \end{aligned} \tag{54}$$

The assumptions about the relative effectiveness of A's technology with no CM versus the opposite situation justify $d_3 > d_4$; as $E[\tau(y)]$ and $E[\min\{\tau(y), \xi(x)\}]$ are convex in y, we conclude that $W_A(x, y)$ is convex in y. It follows that Proposition 1 holds for the modified utility function, establishing the existence of a Nash equilibrium.

Another important area of future research is the extension of the nonzero sum game presented in Section 4 to situations in which part of each party's investment is aimed at developing its technology (or CM), while another part is aimed at disrupting the efforts of the other party to develop its technology (or CM). For example, if D's investment will be given by the pair (x_1, x_2), and A's investment will be given by (y_1, y_2), the parameters for the respective distribution functions of the development times might be of the form: $\mu(x_1, y_2) = a_1 + b_1 x_1 - c_1 y_2$ and $\lambda(y_1, x_2) = a_2 + b_2 y_1 - c_2 x_2$. A somewhat similar analysis, albeit at the strategic (national) level, has recently been carried out by Poveda and Tauman [25]. We expect that such scenarios may lead to interesting findings with (possibly) multiple equilibria.

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