Operations Research Letters 36 (2008) 657–661

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Optimal investment in development projects

Boaz Golany*, Uriel G. Rothblum

Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa 32000, Israel

ARTICLE INFO

Article history:
Received 18 February 2008
Accepted 30 June 2008
Available online 9 August 2008

Keywords:
Development projects
Optimal burn rates
Optimal stopping rules

ABSTRACT

We consider investments in development projects within competitive environments where the "winner takes everything". Under stationary uncertainty, it is optimal to start investing immediately at full capacity and continue until exhausting the allocated budget. For non-stationary environments, active investment, possibly deferred, will always be at full capacity; an example demonstrates that further tractable structure may not be available.

1. Introduction

We consider the problem of determining the optimal rate of investment (a.k.a. "burn-rate") in development projects subject to periodic and global constraints. The effect of the investment is to gradually increase the value of the product which is developed.

We focus here on Development (D) – rather than Research & Development (R&D) – projects where there is no uncertainty in the company's ability to build up the value of the product with appropriate investment, given sufficient time and resources to do so. Uncertainty is present in our model only as an exogenous factor—while the company develops its product, there is always the possibility that its competitors will win the "time-to-market" race. If that happens, the D-project is terminated, no revenues are received and all the resources that were invested in it are lost.

We consider finite and infinite horizon scenarios. In the finite horizon case there is a given interval of time during which the company can invest in developing the product. Such situations may occur, for example, when another party (e.g., a governmental agency) announces a due-date when it intends to evaluate competing proposals and select one of them. Another setting which is quite common in some technology markets is the existence of events (e.g., exhibitions) which are set in advance to some future dates and the participating companies are committed to present their latest developments in these events. Infinite horizon situations occur when companies develop products without any specific deadline. In such cases, the development process continues until the company's management determines that the product should be launched into the market or the allocated budget is exhausted. Due to the threat of competitive rivals, this may happen even if it is estimated that the product is not mature enough to be launched but has accumulated sufficient value that makes it attractive to be sold to another company in a transaction which is profitable to both parties.

The models we develop differ from previous work mainly in the treatment of uncertainty when solving simultaneously for the optimal burn-rates and the entry and exit times. While most of the research on dynamic investments in R&D projects focused on single firm optimization problems, our work assumes that competition is present and that it might cause premature termination of the D-project with zero benefits. One of the earliest publications in this area was done by Nobel laureate Robert Lucas, [7]. Lucas considered four possible models where the differences were caused by treating the development time and expenses per unit of time as either fixed or as decision variables. While in some parts of Lucas's paper the completion time of the project is considered as a random variable, the paper does not consider the possible termination of the project (and the consequent loss of all the accumulated investments up to that time) due to external (competition) factor. In fact, one of Lucas's concluding remarks *"If the return varies with time (as would be the case if the firm were in a "race" with competitors) the analysis of controllable costs becomes considerably more difficult"* serves as a partial motivation to the present paper. Kamien and Schwartz [5] derived the optimal timing of planned expenditures in R&D projects in which the total effort (in budget and time) required in order to complete the project successfully is unknown. Roberts and Weitzman, [10], consider various scenarios in which there is uncertainty in the development costs and the benefits that might be incurred upon successful completion of the project. They distinguish between R&D projects in which the firm must complete a certain number of development stages in order to release the product and gain the expected return and other projects (e.g., mineral exploration) in which the project can be terminated at various times and some return will be accrued. Their paper focuses on sequential decision making, where new information is revealed in each stage

* Corresponding author.
E-mail addresses: golany@ie.technion.ac.il (B. Golany), rothblum@ie.technion.ac.il (U.G. Rothblum).

0167-6377/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.orl.2008.06.011

and, in general, the uncertainty in the future costs and benefits decreases as more information is accumulated. However, they do not consider the possibility that the project will be terminated abruptly when a competitor is able to finish a similar project first. Similar non-competitive environments are assumed in later works on stopping rules, e.g., Deshmukh and Chikete [2], Zuckerman [12], Posner and Zuckerman [9], Chi et al. [1].

Intensive work on competitive environments started in the mid 1980s. This line of work includes the analysis of "winner-takes-it-all" situations by Harris and Vickers [3], Park [8], Lippman and McCardle [6] among others. In particular, Spector and Zuckerman [11] considered a situation similar to the one we assume here. They show that under some conditions the burn-rates should increase monotonically as a function of the project state. Jansen, [4], considers a "stealth" race among firms developing similar technologies where each competitor doesn't even know who else is competing, let alone the progress made by other competitors. Jansen explores this competition and reaches a conclusion that contradicts that of Spector and Zuckerman — that under some assumptions the equilibria investment rates should decrease in time.

The paper is organized as follows. In Section 2 we provide the notation and the assumptions underlying our analysis. In Section 3 we explore stationary scenarios (where the probability of premature termination of the D-project doesn't change with time). Section 4 investigates non-stationary scenarios and employs an example to demonstrate some unexpected outcomes that might occur due to the unstructured nature of the optimization problem. Finally, in Section 5 we offer some concluding remarks.

2. Data and decision variables

2.1. Parameters

- τ The random occurrence time of an exogenous factor that would cause termination of our D-project, resulting in the loss of all invested funds (e.g., the random time at which our competitor will complete developing a similar product and launch it to the market).
- $R_\tau(\cdot)$ The cumulative probability distribution function of the random time τ. (Specifically, $R_\tau(t) = \Pr\{\tau \le t\}\ \forall t > 0$ is the probability that a D-project that started at time 0 will be terminated before time t due to the exogenous factor. In particular, $R_\tau(t)$ is increasing, right-continuous, $R_\tau(0) = 0$ and $R_\tau(t) \to 1$ as $t \to \infty$.)
- $W(\cdot)$ The potential value gained from launching the product into the market (at any time instance) as a function of the cumulative amount that has already been invested in it, provided that the development process will not be terminated prematurely. We assume throughout that the function $W(\cdot)$ is concave, increasing, bounded from above and satisfies $W(0) = 0$.
- C A finite upper bound on the burn-rate (e.g., a cash flow constraint) that applies at any time instance during the development process.
- T An upper bound on the "time to market", $0 \le T \le \infty$.
- B Total available budget, $0 \le B \le \infty$.
- x Amount invested to date.

2.2. Decision variables and optimal objective value

- $\widehat{T}$ The planned "time to market", restricted by $0 \le \widehat{T} \le T$.
- $\widehat{B}$ The allocated budget, restricted by $0 \le \widehat{B} \le B - x$.
- $\widehat{b}_t$ The burn rate at time $0 \le t \le \widehat{T}$, assuming exogenous termination has not occurred before that time. The burn rates must satisfy $0 \le \widehat{b}_t \le C$ and $\int_0^{\widehat{T}} \widehat{b}_t \mathrm{d}t = \widehat{B}$.
- $\widehat{V}(x)$ The optimal expected profit from the project after a cumulative amount x has already been invested in developing the product. This value accounts for the potential value of marketing the project in the future (assuming that this happens before a competitor forces termination) and for the future investments cost.

We distinguish between stationary and non-stationary analysis. In the stationary case, we assume that the random exogenous factor has a memoryless distribution (e.g., no information is gained about future progress in the competitors' effort if at a particular time they are not yet ready). This assumption implies that τ has an exponential distribution, that is, for some $\lambda > 0$, $1 - R_\tau(t) = \mathrm{e}^{-\lambda t}$ for every $t > 0$. This reflects a situation where we gain no information about our competitors, except if and when they announce a victory in the race. In contrast, in the non-stationary scenario, some information is gained as time goes by. Here, we will assume that the external termination has an arbitrary distribution whose cumulative distribution function $R_\tau(\cdot)$ is arbitrary (rather than the memoryless exponential distribution which is assumed in the stationary scenario). In such cases, the stochastic information that is available to the decision maker is non-stationary. Specifically, if termination has not occurred by time s, the probability that termination will occur after time $t + s$ is given by the conditional probability $\Pr\{\tau > t + s | \tau > s\} = \frac{\Pr\{\tau > t+s\}}{\Pr\{\tau > s\}} = \frac{1-R_\tau(t+s)}{1-R_\tau(s)}$; in the non-exponential case, this expression is different from $\Pr\{\tau > s\}$.

3. Analysis of the stationary case

The expected profit from the D-project after investing a cumulative amount of x, assuming that external termination has not yet happened, is expressed by:

$$\widehat{V}(x) = \max_{\substack{0 \le \widehat{b}_t \le C, 0 \le t \le \widehat{T} \\ \int_0^{\widehat{T}} \widehat{b}_t \mathrm{d}t = \widehat{B} \\ 0 \le \widehat{T} \le T, 0 \le \widehat{B} \le B - x}} \left\{ -\int_0^{\widehat{T}} \mathrm{e}^{-\lambda t} \widehat{b}_t \mathrm{d}t + \mathrm{e}^{-\lambda \widehat{T}} W\left(x + \widehat{B}\right) \right\}. \quad (1)$$

The first term in the objective function in (1) represents the expected future investment costs. Notice that we will continue to invest only if the project is not terminated and this is expressed by multiplying the burn-rate at time t with the probability that the project is still active at that time. The second term represents the expected value of the product at time $\widehat{T}$—the marketing time of the project. Here, again, the value $W(\cdot)$ is multiplied by the appropriate probability. The objective function does not account for the sunk cost x that has already been spent. The maximization in (1) is subject to two constraints—first, the burn-rate at any time can not be larger than the upper bound and second, the cumulative future investments can not go beyond the total available budget.

Consider a feasible solution consisting of $\widehat{T}$, $\widehat{B}$ and $\widehat{b}_t$, for $0 \le t \le \widehat{T}$. Of course, $\widehat{B}$ and $\widehat{T}$ are realizable by the $\widehat{b}_t$'s if and only if $\widehat{T}C \ge \widehat{B}$. It follows that (1) can be rewritten as

$$\widehat{V}(x) = \max_{\substack{0 \le \widehat{B} \le B - x \\ \widehat{B} \le C\widehat{T} \\ \widehat{T} \le T}} \left\{ \max_{\substack{0 \le \widehat{b}_t \le C, 0 \le t \le \widehat{T} \\ \int_0^{\widehat{T}} \widehat{b}_t \mathrm{d}t = \widehat{B}}} \left\{ -\int_0^{\widehat{T}} \mathrm{e}^{-\lambda t} \widehat{b}_t \mathrm{d}t + \mathrm{e}^{-\lambda \widehat{T}} W\left(x + \widehat{B}\right) \right\} \right\}. \quad (2)$$

The internal maximization problem in (2) is parametric, with $\widehat{T}$ and $\widehat{B}$ as constant parameters; as the second term in its objective function is constant, it reduces to

$$-\min_{\substack{0 \le \widehat{b}_t \le C, 0 \le t \le \widehat{T} \\ \int_0^{\widehat{T}} \widehat{b}_t \mathrm{d}t = \widehat{B}}} \left\{ \int_0^{\widehat{T}} \mathrm{e}^{-\lambda t} \widehat{b}_t \mathrm{d}t \right\}. \quad (3)$$

Lemma 1. *The unique optimal solution of the minimization problem (3) is given by*

$$b_t^* = \begin{cases} C & \text{if } \widehat{T} - \frac{\widehat{B}}{C} \le t \le \widehat{T} \\ 0 & \text{if } 0 \le t < \widehat{T} - \frac{\widehat{B}}{C}. \end{cases} \tag{4}$$

Proof. The minimization problem in (3) is a continuous knapsack problem with a standard solution given by (4); uniqueness is guaranteed since $e^{-\lambda t}$ is strictly decreasing in t. □

Given $\widehat{T}$ and $\widehat{B}$, we next substitute (4) into the internal optimization problem of (2) to obtain

$$\begin{aligned} \widehat{V}(x) &= \max_{0\le\widehat{B}\le B-x, \widehat{B}\le C\widehat{T}, \widehat{T}\le T} \left\{ -\int_{\widehat{T}-\frac{\widehat{B}}{C}}^{\widehat{T}} e^{-\lambda t} C \mathrm{d}t + e^{-\lambda\widehat{T}} W(x+\widehat{B}) \right\} \\ &= \max_{0\le\widehat{B}\le B-x, \widehat{B}\le C\widehat{T}, \widehat{T}\le T} e^{-\lambda\widehat{T}} \left\{ \frac{C}{\lambda}(1-e^{\frac{\lambda\widehat{B}}{C}}) + W(x+\widehat{B}) \right\} \\ &= \max_{0\le\widehat{B}\le B-x, \frac{\widehat{B}}{C}\le T} \left\{ \max_{\frac{\widehat{B}}{C}\le\widehat{T}\le T} \left\{ e^{-\lambda\widehat{T}} \left[\frac{C}{\lambda}(1-e^{\frac{\lambda\widehat{B}}{C}}) + W(x+\widehat{B}) \right] \right\} \right\}. \end{aligned} \tag{5}$$

Consider the function $f(\cdot)$ defined for $\widehat{B} \ge 0$ by $f(\widehat{B}) = \frac{C}{\lambda}(1-e^{\frac{\lambda\widehat{B}}{C}}) + W(x+\widehat{B})$; we observe that the function is concave and $f(0) = W(x) \ge 0$. It follows that we can restrict $\widehat{B}$ in (5) to satisfy $f(\widehat{B}) \ge 0$, that is, $\widehat{B} \le \overline{B}$ where $\overline{B} \equiv \sup\{\widehat{B} \ge 0 : f(\widehat{B}) \ge 0\} \le \infty$. With this constraint, the expression multiplying $e^{-\lambda\widehat{T}}$ in (5) is nonnegative and, given $\widehat{B}$, the optimal $\widehat{T}$ for the bracketed optimization problem in (5) is $T^* \equiv \frac{\widehat{B}}{C}$. So, (5) becomes

$$\widehat{V}(x) = \max_{0\le\widehat{B}\le\min\{\overline{B}, B-x, CT\}} \left\{ \frac{C}{\lambda}\left(e^{-\frac{\lambda\widehat{B}}{C}} - 1\right) + W(x+\widehat{B}) \right\}. \tag{6}$$

The optimization problem in (6) has a strictly concave objective function over a closed interval and therefore its solution is simple to characterize. Specifically, the derivative of the objective function given by $W'(x+\widehat{B}) - e^{-\frac{\lambda\widehat{B}}{C}}$ is strictly decreasing. Let

$$B^* \equiv \sup\left\{ 0 \le \widehat{B} \le \min\{B-x, CT\} : W'(x+\widehat{B}) \ge e^{-\frac{\lambda\widehat{B}}{C}} \right\}, \tag{7}$$

with $B^* = 0$ when the supremum is taken over the empty set. Evidently, $B^* \le \overline{B}$ and B^* is the optimal solution of the optimization problem in (6) and the "external" optimization problem in (5). Thus we have proven the following result.

Theorem 1. *The unique optimal solution of the maximization problem in (1) has B^* given by (7), $T^* = \frac{B^*}{C}$ and $b_t^* = C$ for $0 \le t \le T^*$.* □

Remarks. 1. The solution of the investment problem (1), provided in Theorem 1, has the following structure: it first allocates a budget B^* that satisfies the restriction $B^* \le B$ and then implements the allocation by using it at full capacity C from time 0 till its depletion.

2. Consider the generalization of the decision problem that is cast in (1) which allows for (stationary) discounting, where the value of a dollar received at time $t > 0$ is $e^{-\rho t}$. In this case, the only change in (1) will be the replacement of each occurrence of the expression $e^{-\lambda t}$ $(0 \le t \le T)$ by $e^{-\rho t}e^{-\lambda t} = e^{-(\rho+\lambda)t}$. So, Theorem 1 and the analysis leading toward its verification apply with one modification: λ should to be replaced by $\rho + \lambda$.

3. Consider the variant of the original investment problem, or the one discussed in the above remark, where the bounds on the burn-rate variables are non-stationary, given by C_t, for time $t \ge 0$. Thus, the decision variables $\widehat{b}_t$ are constrained by $0 \le \widehat{b}_t \le C_t$. In this case, C_t replaces C in (1) and $\int_0^{\widehat{T}} C_t \mathrm{d}t$ replaces $C\widehat{T}$ in (2). Next, the solution of the continuous knapsack problem in Lemma 1 will satisfy (4) with $\frac{\widehat{B}}{C}$ replaced by the unique Δ^* satisfying $\int_{\widehat{T}-\Delta^*}^{\widehat{T}} C_t \mathrm{d}t = \widehat{B}$ and the same substitution will be used in Theorem 1.

4. The solution of the investment problem given in Theorem 1 provides a complete investment plan, one which does not require any future re-examination and/or update (assuming that there are no changes in the problem parameters). But, as the process evolves, information is revealed about the external termination process (about the position of the competitor), that is, at any time $t > 0$ the decision maker is told whether or not $\tau > t$. We next argue that such information does not give the decision maker any advantage. In principle, such information could potentially be useful only if $\tau > t$ (for otherwise the process terminates before time t and no investment takes place beyond time t). As decisions made at the beginning of the planning horizon for time instances beyond time t are applicable only if $\tau > t$, any improvement made over the optimal solution obtained in Theorem 1 by observing the process a time t could have been made a-priori.

5. Spector and Zuckerman [11] consider an investment problem where premature termination allows the investor to secure a fraction $0 < \gamma < 1$ of the investment cost. For our problem, this situation can be incorporated into (1) by adding to the objective function the term $\int_0^T [\gamma \int_0^t \widehat{b}_s \mathrm{d}s] \lambda e^{-\lambda t} \mathrm{d}t = \gamma \int_0^T e^{-\lambda t} \widehat{b}_t \mathrm{d}t$ (the equality following from integration by parts); thus, the first term of the objective function in (1) is multiplied by $(1-\gamma)$. It then follows that our analysis and solution of the investment problem applies with only one minor change—the replacement of W by $(1-\gamma)^{-1}W$.

4. Analysis of the non-stationary case

Henceforth we shall use the notation $G_\tau(t)$ for $\Pr\{\tau > t\} = 1 - R_\tau(t)$. Of course, $G_\tau(t)$ is non-increasing in t and, assuming continuity at 0, $G_\tau(0) = 1$. The investment problem that the decision maker faces is then cast by a modification of (1) obtained by replacing $e^{-\lambda t}$ by $G_\tau(t)$. The modified optimization problem is then represented by the corresponding modification of (2); further, the proof of Lemma 1 shows that (4) is a solution to the internal optimization problem in the modified version of (2) (and is unique when $G_\tau(\cdot)$ is strictly decreasing). Using (4), we then have that the investment problem reduces to determining the budget and termination time in the following modification of (5)

$$\begin{aligned} \widehat{V}(x) &= \max_{0\le\widehat{B}\le B-x, \widehat{B}\le C\widehat{T}, \widehat{T}\le T} \left\{ -\int_{\widehat{T}-\frac{\widehat{B}}{C}}^{\widehat{T}} G_\tau(t) C \mathrm{d}t + G_\tau(\widehat{T}) W(x+\widehat{B}) \right\} \\ &= \max_{0\le\widehat{B}\le B-x, \widehat{B}\le CT} \left[\max_{\frac{\widehat{B}}{C}\le\widehat{T}\le T} \left\{ -\int_{\widehat{T}-\frac{\widehat{B}}{C}}^{\widehat{T}} G_\tau(t) C \mathrm{d}t \right.\right. \\ &\quad \left.\left. + \; G_\tau(\widehat{T}) W(x+\widehat{B}) \right\} \right]. \end{aligned} \tag{8}$$

The solution of (8) turns out to be more complicated than the solution of (5); still, the five remarks that follow Theorem 1 apply, with one exception—the part of Remark 1 that asserts that investment at the maximal burn-rate is to start at time 0. The complexity of the structure and analysis of the non-stationary version of the investment problem (which is reducible to (8)) is illustrated via the following example.

Example. Consider the parametric non-stationary optimal investment problem given in (8) with $C = 1, x = 0, T = \infty$,

$$G_\tau(t) = \begin{cases} 1-t & \text{if } t \le \frac{1}{2} \\ \frac{1}{2} & \text{if } t > \frac{1}{2} \end{cases} \quad \text{and} \quad W(\widehat{B}) = \begin{cases} 2\widehat{B} & \text{if } \widehat{B} \le 10 \\ 20 & \text{if } \widehat{B} > 10, \end{cases} \tag{9}$$

with B as a parameter. Note that in this example, $R_\tau(t) = 1 - G_\tau(t)$ is a deficient distribution, putting positive probability mass at ∞ (an abnormality that is later relaxed); still, Lemma 1 applies. Let $F(\widehat{B})$ represent the optimal value of the bracketed optimization problem in (8), that is,

$$F(\widehat{B}) \equiv \max_{\widehat{T} \ge \widehat{B}} \left\{ -\int_{\widehat{T}-\widehat{B}}^{\widehat{T}} G_\tau(t)\mathrm{d}t + G_\tau(\widehat{T})W(\widehat{B}) \right\}, \tag{10}$$

also, let $f(\cdot|\widehat{B})$ denote the objective function in (10), defined over the domain $\widehat{T} \ge \widehat{B}$. We next determine the function $F(\cdot)$ for each $\widehat{B} \ge 0$ by considering three regions.

Region I: $\widehat{B} > 10$. In this region we further distinguish between two cases.

i. If $\widehat{T} - \widehat{B} \ge \frac{1}{2}$ (that is, investment starts after time $\frac{1}{2}$), then,

$$f(\widehat{T}|\widehat{B}) = -\frac{1}{2}\widehat{B} + 10$$

ii. If $(10 \le)\widehat{B} \le \widehat{T} < \frac{1}{2} + \widehat{B}$ (i.e., investment starts at some point in time before $\frac{1}{2}$), then
(as $1 - t > \frac{1}{2}$ for $t < \frac{1}{2}$, and $G_\tau(t)$ appears with a negative sign in the first term in (10)), we get

$$f(\widehat{T}|\widehat{B}) < -\frac{1}{2}\widehat{B} + 10.$$

So, in this region $F(\widehat{B}) = -\frac{1}{2}\widehat{B} + 10$ and an optimal starting time is any time after $\frac{1}{2}$.

Region II: $\frac{1}{2} \le \widehat{B} \le 10$. First, we note that the feasibility requirement $\widehat{T} \ge \widehat{B}$ implies $\widehat{T} \ge \frac{1}{2}$.
Next, we distinguish again between two cases:

i. If $\widehat{T} > \widehat{B} + \frac{1}{2}$, then

$$f(\widehat{T}|\widehat{B}) = -\frac{1}{2}\widehat{B} + \frac{1}{2}2\widehat{B} = \frac{1}{2}\widehat{B}.$$

ii. If $(\frac{1}{2} \le)\widehat{B} \le \widehat{T} < \widehat{B} + \frac{1}{2}$, then

$$f(\widehat{T}|\widehat{B}) = -\int_{\widehat{T}-\widehat{B}}^{\widehat{T}} G_\tau(t)\mathrm{d}t + G_\tau(\widehat{T})W(\widehat{B}) < -\frac{1}{2}\widehat{B} + \frac{1}{2}2\widehat{B} = \frac{1}{2}\widehat{B}.$$

A minimizer for (10) is any value $\widehat{T} \ge \widehat{B} + \frac{1}{2}$, implying that in this region $F(\widehat{B}) = \frac{1}{2}\widehat{B}$ and an optimal starting time is again any time after $\frac{1}{2}$.

Region III: $\widehat{B} < \frac{1}{2}$. Here, we need to distinguish among three cases:

i. If $\widehat{T} \ge \widehat{B} + \frac{1}{2}$ then

$$f(\widehat{T}|\widehat{B}) = -\frac{1}{2}\widehat{B} + \widehat{B} = \frac{1}{2}\widehat{B}.$$

ii. If $\frac{1}{2} \le \widehat{T} < \frac{1}{2} + \widehat{B}$ then

$$f(\widehat{T}|\widehat{B}) = -\int_{\widehat{T}-\widehat{B}}^{\widehat{T}} G_\tau(t)\mathrm{d}t + G_\tau(\widehat{T})W(\widehat{B}) < \frac{1}{2}\widehat{B}$$

(specifically, in this case $f(\widehat{T}|\widehat{B}) = -\frac{1}{8} - \frac{\widehat{B}^2}{2} + \widehat{T}(\frac{1}{2} + \widehat{B}) - \frac{\widehat{T}^2}{2}$).

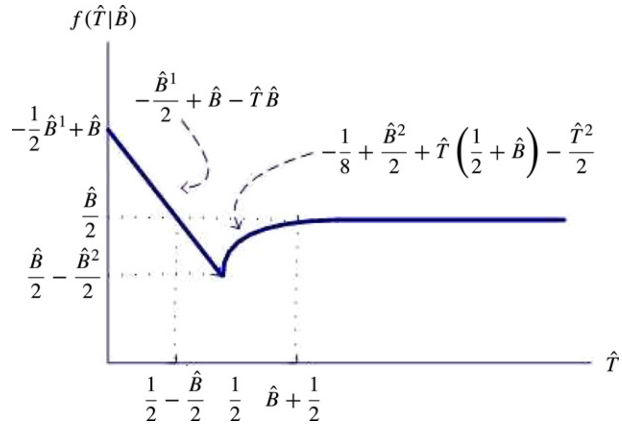

Fig. 1. $f(\widehat{T}|\widehat{B})$ as a function of $\widehat{T}$ for $\widehat{B} < \frac{1}{2}$.

iii. If $\widehat{T} < \frac{1}{2}$, then

$$\begin{aligned} f(\widehat{T}|\widehat{B}) &= -\int_{\widehat{T}-\widehat{B}}^{\widehat{T}} G_\tau(t)\mathrm{d}t + G_\tau(\widehat{T})W(\widehat{B}) \\ &= -\int_{\widehat{T}-\widehat{B}}^{\widehat{T}} (1-t)\mathrm{d}t + (1-\widehat{T})W(\widehat{B}) \\ &= \frac{1}{2}(1-t)^2 \,|_{\widehat{T}-\widehat{B}}^{\widehat{T}} + 2(1-\widehat{T})\widehat{B} \\ &= \frac{1}{2}[(1-\widehat{T})^2 - (1-\widehat{T}+b)^2] + 2(1-\widehat{T})\widehat{B} \\ &= -(1-\widehat{T})\widehat{B} - \frac{1}{2}\widehat{B}^2 + 2(1-\widehat{T})\widehat{B} \\ &= -\frac{1}{2}\widehat{B}^2 + (1-\widehat{T})\widehat{B}. \end{aligned} \tag{11}$$

To maximize the last expression of (11) we need to select the smallest value of $\widehat{T}$, namely, $\widehat{T} = \widehat{B}$ which yields $f(\widehat{T}|\widehat{B}) = \widehat{B} - \frac{3}{2}\widehat{B}^2$. Thus, we get that for Region III,

$$F(\widehat{B}) = \min\left\{\frac{1}{2}\widehat{B}, \widehat{B} - \frac{3}{2}\widehat{B}^2\right\} = \begin{cases} \widehat{B} - \frac{3}{2}\widehat{B}^2 & \text{if } \widehat{B} \le \frac{1}{3} \\ \frac{1}{2}\widehat{B} & \text{if } \frac{1}{3} \le \widehat{B} < \frac{1}{2}. \end{cases} \tag{12}$$

Fig. 1 demonstrates $f(\widehat{T}|\widehat{B})$ as a function of $\widehat{T}$ for $\widehat{B} < \frac{1}{2}$ (ignoring the restriction $\widehat{T} \ge \widehat{B}$).

The above analysis leads to an interesting conclusion — when $\widehat{B} < \frac{1}{3}$ the unique optimal starting time is 0. However, when $\widehat{B} > \frac{1}{3}$ it is optimal to start investing only after time $\frac{1}{2}$!!. We note that the maximizers of the function $f(\cdot|\frac{1}{3})$ are $\widehat{T}^* \in \{\frac{1}{3}\} \cup [\frac{5}{6}, \infty)$, a set which is not convex! So, optimal starting time when $\widehat{B} = \frac{1}{3}$ is either 0 or any time after $\frac{1}{2}$, but no other time! We emphasize that the function $f(\cdot|\frac{1}{3})$ is neither increasing, nor decreasing, nor convex, nor concave; furthermore, its maximizing set is not convex.

The analysis of the three regions of $\widehat{B}$ leads to the following conclusion:

$$F(\widehat{B}) = \begin{cases} \widehat{B} - \frac{3}{2}\widehat{B}^2 & \text{if } \widehat{B} \le \frac{1}{3} \\ \frac{1}{2}\widehat{B} & \text{if } \frac{1}{3} \le \widehat{B} \le 10 \\ -\frac{1}{2}\widehat{B} + 10 & \text{if } 10 < \widehat{B}. \end{cases} \tag{13}$$

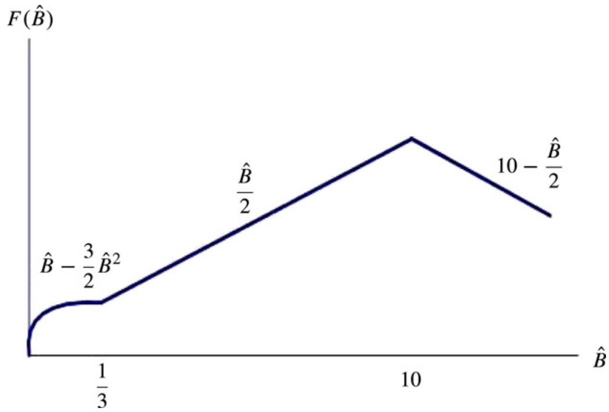

Fig. 2. $F(\widehat{B})$ as a function of $\widehat{B}$.

As seen in Fig. 2, the function $F(\cdot)$ is neither convex nor concave, but it is unimodal with a maximum attained at $\widehat{B} = 10$. It follows that with budget B, the optimal use of the budget will be $\min\{B, 10\}$ and the optimal starting time for investment will be 0 if $B \le \frac{1}{3}$ and any time past $\frac{1}{2}$ if $B \ge \frac{1}{3}$ (and no other starting times are optimal).

Variant of the example Consider a variant of the above example where the function $G_\tau(t)$ in (9) is multiplied by $e^{-\rho t}$, with ρ as a small positive number (e.g., the introduction of discounting with ρ as the interest rate). The resulting function $f(\cdot|\widehat{B})$ and $F(\cdot)$ will then be slightly perturbed and the optimal starting time for $\widehat{B} > \frac{1}{3}$ will be unique. In fact, at time $s > \frac{1}{2}$, the conditional distribution of future survival given survival till time s, equals $\Pr(\tau > t+s|\tau > s) = \frac{G_\tau(t+s)}{G_\tau(s)} = e^{-\rho t}$. So, the problem that the decision maker will face at time s, assuming that external termination does not occur beforehand, belongs to the class of stationary problems for which the existence of unique solution was demonstrated in Section 3 (in particular, investment should start instantly). The unique optimal solution of starting early will be preserved for a slight perturbation of the region $\widehat{B} < \frac{1}{2}$.

Interpretation for the solution of the example (and its variant) Under the example, for any $s \ge \frac{1}{2}$ we have that $\Pr(t > s + t \mid t > s) = 1$ for each $t \ge 0$. Thus, whenever time $s \ge \frac{1}{2}$ is reached without the competitor completing its project, the investor is guaranteed to realize, without risk, the value of the project at any time he decides to realize (in order to realize the full value this should happen after an investment of 10 which will yield the full potential value of 20). So, money that is invested after time $\frac{1}{2}$ is not at risk. It follows that in this example, the decision of the starting time of the project has to balance between starting early with increased risk of losing the invested funds with decreased risk of not materializing the value of the project versus starting late (at time $\frac{1}{2}$) with the opposite trends. The optimal solution we obtain shows that when the budget is large (above $\frac{1}{3}$), it is best to eliminate the risk of the investment and start late, and when the budget is small, it is best to start early and minimize the risk of not realizing the project. Similar considerations apply to the variant of the example.

5. Concluding remarks

The result we obtained for the non-stationary example (that there exist situations in which it is better to wait even though we are in a winner-takes-all race) may seem surprising. Still, it is supported by evidence from various real-world cases (e.g., where smaller firms wait for some time during which they believe that a larger firm is investing in some D-project; after a certain time elapses without any announcement from the larger firm, they start their own investment).

In future research we intend to further investigate some game-theoretic aspects of this problem. For example, if the market contains two symmetric firms and both operate according to the result we found in Section 4, none of them will start investing at time 0. Some mechanism might be required to break the "deadlock" otherwise they may end up never investing.

Acknowledgment

This research was partially supported by the Center for Security Science and Technology at the Technion.

References

[1] T.L. Chi, J. Liu, H. Chen, Optimal stopping rule for a project with uncertain completion time and partial salvageability, IEEE Transactions on Engineering Management 44 (1) (1997) 54–66.
[2] S. Deshmukh, S. Chikete, Dynamic investment strategies for a risky R&D project, Journal of Applied Probability 14 (1) (1977) 144–152.
[3] C. Harris, J. Vickers, Perfect equilibrium in a model of race, Review of Economic Studies 52 (1985) 193–209.
[4] J. Jansen, Innovating under the radar: Investment dynamics under asymmetric information, Working Paper, Wissenschaftszentrum Berlin (WZB), Germany, 2003.
[5] M. Kamien, N. Schwartz, Expenditure patterns for risky R&D projects, Journal of Applied Probability 8 (1) (1971) 60–73.
[6] S.A. Lippman, K.F. McCardle, Dropout behavior in R&D race with learning, Rand Journal of Economics 18 (1987) 287–295.
[7] R.E. Lucas, Optimal management of a research and development project, Management Science 17 (11) (1971) 679–697.
[8] J. Park, Dynamic patent races with risky choices, Management Science 33 (12) (1987) 1563–1571.
[9] M. Posner, D. Zuckerman, Optimal R&D programs in a random environment, Journal of Applied Probability 27 (2) (1990) 343–350.
[10] K. Roberts, M.L. Weitzman, Funding criteria for research, development, and exploration projects, Econometrica 49 (5) (1981) 1261–1288.
[11] Y. Spector, D. Zuckerman, Expenditure patterns and timing of patent protection in a competitive R&D environment, Operations Research Letters 20 (1997) 237–242.
[12] D. Zuckerman, A diffusion process model for the optimal investment strategies of an R&D project, Journal of Applied Probability 17 (3) (1980) 646–653.