CONSISTENT WEIGHTS FOR JUDGEMENTS MATRICES OF THE RELATIVE IMPORTANCE OF ALTERNATIVES *

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We prove that the only solution satisfying consistency axioms for the problem of retrieving weights from inconsistent judgements matrices whose entries are the relative importance ratios of alternatives is the geometric mean.

eigenvector * geometric mean * decision analysis * judgement matrices * importance of alternatives

1. Introduction

The framework of the analytic hierarchy process (see Saaty [4]) is a very useful tool for the solution of diverse problems. The basic step in this process is the solution of the following problem. Given a matrix $A = (a_{ij})$ satisfying $0 < a_{ij} = 1/a_{ji}$, of approximate judgements of the ratios of the underlying weights associated with objects i and j, retrieve the weights w_i from A such that w_i/w_j will approximate a_{ij} in some sense. Saaty uses the principal eigenvector (the eigenvector corresponding to the largest eigenvalue) of A to solve this problem. He then combines the solutions of such subproblems using arithmetic means to compute a single weight vector. In case of more than one decision maker, the geometric mean of the weight vectors of individual decision makers is formed to produce a single weight vector corresponding to the consensus judgement.

Saaty's rationale for using the eigenvector solution is that it generates the correct solution in the consistent case (namely, when $a_{ij} = w_i/w_j$ for some weight vector w), and is close to it if A is close to being consistent, i.e., it is continuous. Obviously, there exists infinitely many solutions with these properties. Moreover, there seems to be no justification for using so disparate solutions for these subproblems.

Since the essence of the problem is resolving inconsistency, one expects the solution not to violate our basic sense of consistency. This, unfortunately, is not the case. The solution generated by the eigenvector method depends on the description of the problem. Indeed, suppose a_{ij} describes by how much object i is *better* than object j. If instead we ask by how much object i is *worse* than object j, the answer should be $b_{ij} = 1/a_{ij}$, and the weights associated with B should be the inverses of those associated with A. Clearly, $B = 1/A$ (componentwise). But if x is the principal eigenvector of A, $1/x$ (componentwise) is not the principal eigenvector of $1/A$. That is, the eigenvector solution is not independent of the description of the problem. We feel that this is sufficient

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reason to reject the eigenvector solution, though there are many other reasons why the eigenvector is not a satisfactory solution. Since $a_{ij} = 1/a_{ji}$ implies $1/A = A^T$, this phenomenon was referred to by Johnson et al. [3] as the right–left asymmetry of the eigenvector solution. It is the dependence of the solution on the description of the problem that is the key to finding the correct solution, rather than its asymmetry (or non-uniqueness).

Furthermore, the weighted arithmetic mean suffers from a different type of inconsistency. It is easy to see that when partial solutions (i.e., levels in the hierarchy) are combined, we get different solutions depending on whether we combine the judgement matrices to one matrix and compute the solution for this matrix, or we solve separately for each matrix and combine the solutions. We will refer to this property as the inter-level inconsistency of Saaty's solution.

Crawford and Williams [2], derive the geometric mean from statistical considerations and show that it is preferable to the eigenvector solution in several important respects. In this paper we formulate three fundamental consistency axioms. We then prove the existence of a unique solution satisfying these axioms (the geometric mean). Thus, it turns out that when one imposes logical conditions on the desired properties of the solution, the geometric mean is the *only* solution to the problem.

2. Structure and notation

The following notation will be used throughout this paper:

$A = (a_{ij})$ is any $n \times n$ matrix satisfying $0 < a_{ij} = 1/a_{ji}$,

$w = (w_k)$ is an n-dimensional vector satisfying

$$w_k > 0, \quad \prod_{k=1}^{n} w_k = 1, \tag{1}$$

$V = (v_{ij})$ is any $n \times n$ matrix satisfying $v_{ij} = w_i/w_j$ for some vector w,

A^*, V^* and w^* are the sets of all A, V and w's, respectively,

f: $A^* \to w^*$ maps a matrix A into a solution w, i.e., $w = f(A)$ is the solution corresponding to A.

Some normalization of the vectors w is needed if the solution is to be unique. The multiplicative normalization (1) is natural as the data are given in terms of ratios, and the fundamental nature of the problem is multiplicative. Indeed, A^*, V^* and w^* are all groups under *componentwise* multiplication, V^* is isomorphic to w^*, and is a subgroup of A^*. We refer the reader to Birkhoff and MacLane [1] for the elementary results from group theory we need. In view of this structure, it is not surprising that the requirement that f be a homomorphism has a natural interpretation in the context of the problem as we shall see below.

The following decomposition of the matrix A will be needed. $A = \prod_{i<j} A^{ij}$ where the matrix A^{ij} is defined by

$$a_{kl}^{ij} = \begin{cases} a_{ij} = 1/a_{lk}^{ij}, & k = i \text{ and } l = j, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, we want to reiterate that all vector and matrix multiplications and divisions are componentwise.

3. The axioms

Axiom 1. *If $A = (w_i/w_j)$ is consistent, the solution is the vector w.*

Axiom 2. *If B corresponds to the same judgements as A except for a permutation of the order of objects, then $f(B)$ should yield the same weights as $f(A)$ except for (the same) permutation of the order of objects.*

Axiom 3. *The mapping f is a homomorphism from A^* to w^*, i.e., $f(AB) = f(A)f(B)$.*

Before turning to a discussion of the meaning of the axioms, we present our main result:

Fundamental theorem. *There is exactly one mapping $w = f(A)$ satisfying Axioms 1–3. It is defined by $w_i = (\prod_{j=1}^{n} a_{ij})^{1/n}$.*

Discussion. A reasonable solution to our problem should have the following three properties:

(I) If $A = (w_i/w_j)$ is consistent, the solution is the vector w.

(II) The solution does not depend on the description of the problem.

(III) The solution leads to inter-level consistent decisions.

Property (I) is of course Axiom 1. The solution should retrieve the correct weights in the consistent case.

Property (II) means that the solution must satisfy Axiom 2 as well as inverse ratio independence of description as discussed above.

Property (III) means that when levels in the hierarchy are combined, we get the same solution whether we combine the judgement matrices to one matrix and compute the solution for this matrix, or we solve separately for each matrix and combine the solutions.

The rationale for Axiom 3 can now be seen. The data are given in multiplicative (ratio) form. The problem has a multiplicative group structure. The requirement that f be a homomorphism preserves this structure. Also, if A and B are consistent, Axiom 1 implies Axiom 3. More importantly, however, our solution – the geometric mean – satisfies

$$f(A_1^{\alpha_1}A_2^{\alpha_2}\cdots A_r^{\alpha_r}) = [f(A_1)]^{\alpha_1}[f(A_2)]^{\alpha_2}\cdots[f(A_r)]^{\alpha_r},$$

which guaranties inter-level consistency, i.e., Property (III). Furthermore, Lemma 1 below shows that Axiom 3 also implies inverse ratio independence of description.

Lemma 1. *Axiom* 3 *implies* $f(1/A) = 1/f(A)$.

Proof. This is an elementary property of homomorphisms: the image of the inverse of A is the inverse of the image of A. □

Note also that f, being a homomorphism, maps the identity of A^* onto the identity of w^*, and that these identities are the matrix and vector of 1's, respectively.

We now show that all three axioms are needed.

Theorem 1. *Axioms 1–3 are independent.*

Proof. The mapping

$$w_i = 1\Big/\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}$$

satisfies Axioms 2 and 3, but not 1.

The mapping

$$w_i = a_{i1}\Big/\left(\prod_{i=1}^{n} a_{i1}\right)^{1/n}$$

satisfies Axioms 1 and 3, but not 2.

The eigenvector solution satisfies Axioms 1 and 2, but not 3. □

4. Proof of fundamental theorem

The proof is constructive. We will show that if f satisfies Axioms 1–3, it must satisfy $w_i = (\prod_{j=1}^{n} a_{ij})^{1/n}$. We need the following results.

Lemma 2. *For the special case* $A = A^{ij}$, $w = f(A)$ *is of the form*

$$w_k = \begin{cases} \alpha, & k = i, \\ 1/\alpha, & k = j, \\ 1 & \text{otherwise}, \end{cases}$$

for some $\alpha > 0$.

Proof. Define B as the matrix corresponding to the same judgements as A^{ij}, but with objects i, j, transposed, and $u = f(B)$. By Axiom 2 we have

$$u_k = \begin{cases} w_j, & k = i, \\ w_i, & k = j, \\ w_k & \text{otherwise}. \end{cases} \tag{2}$$

However, $B = 1/A$. This, by Axiom 3, implies $u_k = 1/w_k$ which, combined with (2) implies $w_j = 1/w_i$, and $w_k = 1$ for $k \neq i$, $k \neq j$. □

Let $A = A^{ij}$, and denote $a = a_{ij}$. Define g: $R^+ \to R^+$ by $g(a) = \alpha$ with α defined in Lemma 2. Note that $g(1) = 1$ since f maps identity to identity.

Lemma 3. *The function g satisfies* $g(1/a) = 1/g(a)$.

Proof. This follows immediately from Lemma 1. □

Lemma 4. *For the special case* $A = A^{ij}$, $w = f(A)$ *is of the form*

$$w_k = \begin{cases} g(a_{ij}), & k = i, \\ g(a_{ji}), & k = j, \\ 1 & \text{otherwise}. \end{cases}$$

Proof. This follows from Lemmas 2 and 3. □

Lemma 5. *The function g satisfies* $g(ab) = g(a)g(b)$.

Proof. Define B by $b_{kl} = 1/b_{lk} = b$ for $k = i$ and $l = j$, and $b_{kl} = 1$ otherwise. The lemma now follows from Axiom 3. □

Lemma 6. *For any A,* $w = f(A)$ *satisfies*

$$w_k = \prod_{j=1}^{n} g(a_{kj}).$$

Proof. Since $A = \prod_{i<j} A^{ij}$, and $g(1) = 1$, we have by Axiom 3

$$w_k = \prod_{j \neq k} g(a_{kj}) = \prod_{j=1}^{n} g(a_{kj}). \quad \square$$

Lemma 7. *The function g satisfies* $g(a) = a^{1/n}$.

Proof. In the special case $a_{ij} = w_i/w_j$ we have from Lemmas 6 and 5 and the normalization (1)

$$w_k = \prod_{j=1}^{n} g(w_k/w_j) = g\left(\prod_{j=1}^{n} w_k/w_j\right)$$

$$= g\left(w_k^n \prod_{j=1}^{n} 1/w_j\right) = g(w_k^n).$$

Since w_k is arbitrary the result follows. □

The proof of the fundamental theorem now follows from Lemmas 6 and 7.

References

[1] G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, 3rd ed., Macmillan, New York, 1965.

[2] G. Crawford and C. Williams, "A note on the analysis of subjective judgement matrices", *J. of Mathematical Psychology* **29**, 387–405 (1985).

[3] C.R. Johnson, W.B. Beine and T.J. Wang, "A note on right-left asymmetry in an eigenvector ranking procedure", *J. of Mathematical Psychology* **19**, 61–64 (1979).

[4] T.L. Saaty, *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.