CONSISTENT WEIGHTS FOR JUDGEMENTS MATRICES OF THE RELATIVE IMPORTANCE OF ALTERNATIVES *

J. BARZILAI

School of Business Administration, Dalhousie University, 6152 Coburg Road, Halifax, N.S., Canada B3H 1Z5

W.D. COOK

Faculty of Administrative Studies, York University, 4700 Keele Street, Downsview, Ont., Canada M3J 2R6

B. GOLANY

Center for Cybernetic Studies, The University of Texas, Austin, TX 78712, USA

Received May 1986
Revised April 1987

We prove that the only solution satisfying consistency axioms for the problem of retrieving weights from inconsistent judgements matrices whose entries are the relative importance ratios of alternatives is the geometric mean.

eigenvector • geometric mean • decision analysis • judgement matrices • importance of alternatives

1. Introduction

The framework of the analytic hierarchy process (see Saaty [4]) is a very useful tool for the solution of diverse problems. The basic step in this process is the solution of the following problem. Given a matrix $A = (a_{ij})$ satisfying $0 < a_{ij} = 1/a_{ji}$, of approximate judgements of the ratios of the underlying weights associated with objects $i$ and $j$, retrieve the weights $w_i$ from $A$ such that $w_i/w_j$ will approximate $a_{ij}$ in some sense. Saaty uses the principal eigenvector (the eigenvector corresponding to the largest eigenvalue) of $A$ to solve this problem. He then combines the solutions of such subproblems using arithmetic means to compute a single weight vector. In case of more than one decision maker, the geometric mean of the weight vectors of individual decision makers is formed to produce a single weight vector corresponding to the consensus judgement.

Saaty's rationale for using the eigenvector solution is that it generates the correct solution in the consistent case (namely, when $a_{ij} = w_i/w_j$ for some weight vector $w$), and is close to it if $A$ is close to being consistent, i.e., is continuous. Obviously, there exists infinitely many solutions with these properties. Moreover, there seems to be no justification for using so disparate solutions for these subproblems.

Since the essence of the problem is resolving inconsistency, one expects the solution not to violate our basic sense of consistency. This, unfortunately, is not the case. The solution generated by the eigenvector method depends on the description of the problem. Indeed, suppose $a_{ij}$ describes by how much object $i$ is better than object $j$. If instead we ask by how much object $i$ is worse than object $j$, the answer should be $b_{ij} = 1/a_{ij}$, and the weights associated with $B$ should be the inverses of those associated with $A$. Clearly, $B = 1/A$ (componentwise). But if $x$ is the principal eigenvector of $A$, $1/x$ (componentwise) is not the principal eigenvector of $1/A$. That is, the eigenvector solution is not independent of the description of the problem. We feel that this is sufficient

* Research supported in part by NSERC Canada grants, nos. A-8802 and A-8966.
reason to reject the eigenvector solution, though
there are many other reasons why the eigenvector
is not a satisfactory solution. Since \( a_{ij} = 1/a_{ji} \)
implies \( 1/A = A^T \), this phenomenon was referred
to by Johnson et al. [3] as the right–left asymme-
try of the eigenvector solution. It is the depend-
ence of the solution on the description of the
problem that is the key to finding the correct
solution, rather than its asymmetry (or non-
uniqueness).

Furthermore, the weighted arithmetic mean
suffers from a different type of inconsistency. It is
easy to see that when partial solutions (i.e., levels
in the hierarchy) are combined, we get different
solutions depending on whether we combine the
judgement matrices to one matrix and compute the
solution for this matrix, or we solve separately
for each matrix and combine the solutions. We
will refer to this property as the inter-level inconsist-
sistency of Saaty’s solution.

Crawford and Williams [2], derive the geometric
mean from statistical considerations and show that
it is preferable to the eigenvector solution in
several important respects. In this paper we for-
mulate three fundamental consistency axioms. We
then prove the existence of a unique solution
satisfying these axioms (the geometric mean).
Thus, it turns out that when one imposes logical
conditions on the desired properties of the solu-
tion, the geometric mean is the only solution to
the problem.

2. Structure and notation

The following notation will be used throughout
this paper:
- \( A = (a_{ij}) \) is any \( n \times n \) matrix satisfying \( 0 < a_{ij} = \frac{1}{a_{ji}} \),
- \( w = (w_k) \) is an \( n \)-dimensional vector satisfying
  \[ w_k > 0, \quad \prod_{k=1}^{n} w_k = 1, \]  \( w \) (1)
- \( V = (v_{ij}) \) is any \( n \times n \) matrix satisfying \( v_{ij} = w_i/w_j \)
  for some vector \( w \),
- \( A^*, V^* \) and \( w^* \) are the sets of all \( A, V \) and \( w \)'s,
  respectively,
- \( f: A^* \to w^* \) maps a matrix \( A \) into a solution \( w \),
  i.e., \( w = f(A) \) is the solution corresponding to \( A \).

Some normalization of the vectors \( w \) is needed
if the solution is to be unique. The multiplicative
normalization (1) is natural as the data are given
in terms of ratios, and the fundamental nature of
the problem is multiplicative. Indeed, \( A^*, V^* \)
and \( w^* \) are all groups under componentwise
multiplication, \( V^* \) is isomorphic to \( w^* \), and is a sub-
group of \( A^* \). We refer the reader to Birkhoff and
MacLane [1] for the elementary results from group
theory we need. In view of this structure, it is not
surprising that the requirement that \( f \) be a homo-
morphism has a natural interpretation in the con-
text of the problem as we shall see below.

The following decomposition of the matrix \( A \)
will be needed. \( A = \prod_{i<j} A^{ij} \) where the matrix
\( A^{ij} \) is defined by

\[ a_{kl}^{ij} = \begin{cases} a_{ij} & k = i \text{ and } l = j, \\ 1 & \text{otherwise.} \end{cases} \]

Finally, we want to reiterate that all vector and
matrix multiplications and divisions are component-
wise.

3. The axioms

Axiom 1. If \( A = (w_i/w_j) \) is consistent, the solution
is the vector \( w \).

Axiom 2. If \( B \) corresponds to the same judgements
as \( A \) except for a permutation of the order of objects,
then \( f(B) \) should yield the same weights as \( f(A) \)
except for (the same) permutation of the order of
objects.

Axiom 3. The mapping \( f \) is a homomorphism from
\( A^* \) to \( w^* \), i.e., \( f(AB) = f(A)f(B) \).

Before turning to a discussion of the meaning of
the axioms, we present our main result:

Fundamental theorem. There is exactly one map-
ing \( w = f(A) \) satisfying Axioms 1–3. It is defined by
\( w_i = (1/n) \prod_{j=1}^{n} a_{ij} \).

Discussion. A reasonable solution to our problem
should have the following three properties:

(I) If \( A = (w_i/w_j) \) is consistent, the solution is
the vector \( w \).

(II) The solution does not depend on the descrip-
tion of the problem.

(III) The solution leads to inter-level consistent
decisions.
Property (I) is of course Axiom 1. The solution should retrieve the correct weights in the consistent case.

Property (II) means that the solution must satisfy Axiom 2 as well as inverse ratio independence of description as discussed above.

Property (III) means that when levels in the hierarchy are combined, we get the same solution whether we combine the judgement matrices to one matrix and compute the solution for this matrix, or we solve separately for each matrix and combine the solutions.

The rationale for Axiom 3 can now be seen. The data are given in multiplicative (ratio) form. The problem has a multiplicative group structure. The requirement that \( f \) be a homomorphism preserves this structure. Also, if \( A \) and \( B \) are consistent, Axiom 1 implies Axiom 3. More importantly, however, our solution – the geometric mean – satisfies

\[
\begin{align*}
    f(A_1^{a_1}A_2^{a_2} \cdots A_n^{a_n}) &= [f(A_1)]^{a_1}[f(A_2)]^{a_2} \cdots [f(A_n)]^{a_n},
\end{align*}
\]

which guarantees inter-level consistency, i.e., Property (III). Furthermore, Lemma 1 below shows that Axiom 3 also implies inverse ratio independence of description.

**Lemma 1.** Axiom 3 implies \( f(1/A) = 1/f(A) \).

**Proof.** This is an elementary property of homomorphisms: the image of the inverse of \( A \) is the inverse of the image of \( A \). \( \square \)

Note also that \( f \), being a homomorphism, maps the identity of \( A^* \) onto the identity of \( w^* \), and that these identities are the matrix and vector of 1's, respectively.

We now show that all three axioms are needed.

**Theorem 1.** Axioms 1–3 are independent.

**Proof.** The mapping

\[
w_i = 1 \left( \prod_{j=1}^{n} a_{ij} \right)^{1/a_i}
\]

satisfies Axioms 2 and 3, but not 1.

The mapping

\[
w_i = a_{ii} \left( \prod_{i=1}^{n} a_{ii} \right)^{1/a_i}
\]

satisfies Axioms 1 and 3, but not 2.

The eigenvector solution satisfies Axioms 1 and 2, but not 3. \( \square \)

4. Proof of fundamental theorem

The proof is constructive. We will show that if \( f \) satisfies Axioms 1–3, it must satisfy \( w_i = (\prod_{j=1}^{n} a_{ij})^{1/a} \). We need the following results.

**Lemma 2.** For the special case \( A = A^{ij} \), \( w = f(A) \) is of the form

\[
w_k = \begin{cases} 
    \alpha, & k = i, \\
    1/\alpha, & k = j, \\
    1 & \text{otherwise},
\end{cases}
\]

for some \( \alpha > 0 \).

**Proof.** Define \( B \) as the matrix corresponding to the same judgements as \( A^{ij} \), but with objects \( i, j \), transposed, and \( u = f(B) \). By Axiom 2 we have

\[
u_k = \begin{cases} 
    w_j, & k = i, \\
    w_i, & k = j, \\
    w_k & \text{otherwise},
\end{cases}
\]

However, \( B = 1/A \). This, by Axiom 3, implies \( u_k = 1/w_k \) which, combined with (2) implies \( w_j = 1/w_j \), and \( w_k = 1 \) for \( k \neq i, k \neq j \). \( \square \)

Let \( A = A^{ij} \), and denote \( a = a_{ij} \). Define \( g: R^+ \rightarrow R^+ \) by \( g(a) = \alpha \) with \( \alpha \) defined in Lemma 2. Note that \( g(1) = 1 \) since \( f \) maps identity to identity.

**Lemma 3.** The function \( g \) satisfies \( g(1/a) = 1/g(a) \).

**Proof.** This follows immediately from Lemma 1. \( \square \)

**Lemma 4.** For the special case \( A = A^{ij} \), \( w = f(A) \) is of the form

\[
w_k = \begin{cases} 
    g(a_{ij}), & k = i, \\
    g(a_{ji}), & k = j, \\
    1 & \text{otherwise},
\end{cases}
\]

**Proof.** This follows from Lemmas 2 and 3. \( \square \)
Lemma 5. The function $g$ satisfies $g(ab) = g(a)g(b)$.

Proof. Define $B$ by $b_{ki} = 1/b_{ik} = b$ for $k = i$ and $l = j$, and $b_{ki} = 1$ otherwise. The lemma now follows from Axiom 3. □

Lemma 6. For any $A$, $w = f(A)$ satisfies

$$w_k = \prod_{j=1}^{n} g(a_{kj}).$$

Proof. Since $A = \prod_{i<j} A^{ij}$, and $g(1) = 1$, we have by Axiom 3

$$w_k = \prod_{j \neq k} g(a_{kj}) = \prod_{j=1}^{n} g(a_{kj}).$$

□

Lemma 7. The function $g$ satisfies $g(a) = a^{1/n}$.

Proof. In the special case $a_{ij} = w_i/w_j$ we have from Lemmas 6 and 5 and the normalization (1)

$$w_k = \prod_{j=1}^{n} g(w_k/w_j) = g\left(\prod_{j=1}^{n} w_k/w_j\right)$$

$$= g\left(w_k^n \prod_{j=1}^{n} 1/w_j\right) = g(w_k^n).$$

Since $w_k$ is arbitrary the result follows. □

The proof of the fundamental theorem now follows from Lemmas 6 and 7.

References