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Alternate Methods of Treating Factor Weights in DEA

Y ROLL

B GOLANY[1]

Technion–Israel Institute of Technology, Haifa, Israel

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Provisions for controlling factor weights constitute a significant extension of the data envelopment analysis (DEA) methodology, as an effective tool for measuring efficiency. This paper suggests a conceptual framework for the treatment of factor weights in DEA. First, the paper proposes general guidelines for setting bounds on factor weights. Then, it develops and presents alternative methods to limit the range within which these factor weights are allowed to vary. All of these methods involve additional information which is entered into the analysis in the form of constraints, bounds or different objective functions. Finally, the implications of the various approaches is discussed.

Key words—DEA, efficiency, multidimensional scaling

1. INTRODUCTION

IN A PREVIOUS PUBLICATION [9], the authors proposed an application procedure for implementing DEA. The present communication extends that procedure by focusing on the treatment of factor weights in productivity analyses. The ever widening range of DEA applications, see [11], provides ample evidence to the necessity of such an extension. In real world applications, whether in production of service situations, where a measure of relative efficiencies of different decision making units (DMUs) is sought, virtually unconstrained factor weights are usually unaccepted. Likewise, it is usually deemed inappropriate to accord widely differing weights to the same factor, when assessing different DMUs.

The need to exercise some control over factor weights in DEA did not go unnoticed. Recently, a series of possible approaches have been put forward. The general approach—a "cone ratio" model, was suggested in [2] (see Appendix). The cone-ratio model generalizes the original CCR model formulation given in [3], by requiring that values for input and output weights should be restricted within given closed cones. The cone ratio formulation allows a variety of restrictions to be imposed on factor weights through suitable definitions of the bounding cones.

Another approach, developed in [12], implements an "assurance region" principle. By this principle (see Appendix) the weight of one output is used as a basis of comparison for weights of all other outputs. Similarly, the weights of all inputs are compared to the weight of one input. Appropriate ratios between the various weights are estimated on the basis of observations and expert opinion.

A third approach, suggested in [10], uses factor weights which were obtained from unbounded runs of DEA, to set upper and lower limits on weights in the "bounded" formulation (see Appendix). Several other treatments of the factor weights bounding problem have been published, examples being [1], [5], [6] and [7].

The purpose of this paper is to summarize and classify the various approaches for setting

[1]This paper was completed while the second author was a Visiting Scholar at the IC^2 Institute, The University of Texas at Austin, 2815 San Gabriel, Austin, TX 78705-3596, USA.

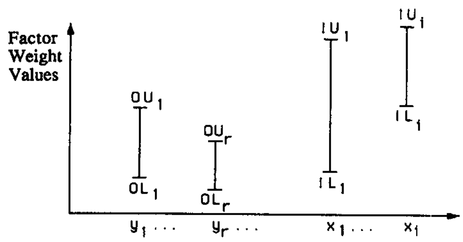

Fig. 1. Absolute ranges for factor weights.

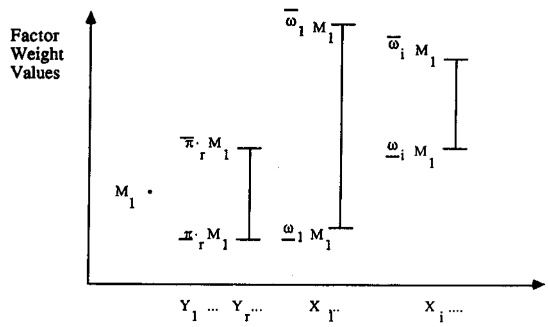

Fig. 2. Relative ranges for factor weights.

bounds on factor weights in DEA, and suggest procedures for determining such bounds. Sections 2 and 3 present the general problem and possible approaches to its solution. Section 4 treats the notion of a common set of weights and examines several ways for defining such a set. Next, a numerical example is worked out, followed by a brief discussion.

2. FACTOR WEIGHTS IN DEA

The DEA efficiency ratio may be presented in the form of a production function as follows:

$$\sum_{r=1}^{R} M_{rj} \cdot y_{rj} = e_j \cdot \left(\sum_{i=1}^{I} N_{ij} \cdot X_{ij} \right) \tag{1}$$

where the M_{rj} and N_{ij} are factor weights (the notation is given in the Appendix).

Factor weights in DEA may be interpreted in various ways. Naturally, in real-world applications, users of the methodology seek to extract meaningful "price" information out of these weights. For example, an interpretation in terms of marginal productivities of the input factors was provided in [3, p. 439]. More recently, a discussion of a case with a single input, multi-output configuration, in which it is easy to relate the (normalized) weights to output prices (per unit of input) was given in [7]. Alternatively, the relative values of the factor weights may occasionally be used for ranking the factors as part of a more general managerial evaluation effort (see [8] and [4]).

A basic difference between the DEA and conventional (additive) production function lies in the flexibility[2] of the factor weights M_{rj} and N_{ij}. In DEA these may take on different values for different DMUs. This flexibility manifests itself in two ways:

(1) No *a priori* values are assigned to the various weights. Thus, in the basic CCR model, the only constraint on factor weights in a positivity requirement (apart from the output–input relationships, providing the relative nature of the analysis)

(2) The same factor may be assigned different weights, when viewing the relative efficiency of different DMUs.

Imposing bounds on factor weights is aimed at controlling both kinds of flexibility. First, the relative "importance" of the various factors is forced to be contained within specified limits. Second, the span of variation of each of the weights is restricted to prescribed ratios.

Factor weights are determined, among other considerations, by the scale on which the specific factors are measured. By making use of the scale-invariance property of the basic DEA model, the obscuring effect of scales may be largely eliminated. This can be done through "normalization" of the observed values in each factor, so that a chosen statistic (e.g. the average) is equal across all factors, in the analyzed set of DMUs. Such multiplicative scale changes enable a clearer comparison of factor importance without affecting the outcomes. Figure 1 depicts schematically a possible deployment of factor weights in a specific application. The notation in the figure above (OU_r, OL_r, IU_i and IL_i) represent the upper and lower bounds on output and input weights, respectively. When numerical values of the different factors are of the same order of magnitude, the position and the spread of ranges within which weights are allowed to vary, to indicate the analyst's attitude towards the various factors. Naturally,

[2]We note that the term flexibility is interpreted differently in conventional econometric and engineering references.

Table 1. Data for numerical example

	Raw data							"Normalized" data						
j	O_1	O_2	O_3	I_1	I_2	I_3	I_4	y_1	y_2	y_3	x_1	x_2	x_3	x_4
1	15500	460	0.85	521	3130	1859	80	103	40	144	97	69	101	112
2	13700	340	0.63	747	5075	3491	44	91	30	107	139	111	190	61
3	18000	1080	0.37	935	1483	2984	93	120	95	63	174	33	162	130
4	8900	490	0.56	205	4583	1736	65	59	43	95	38	101	94	91
5	10800	960	0.14	177	2990	1823	87	72	84	24	33	66	99	121
6	17300	890	0.47	584	5467	1775	98	115	78	79	109	120	97	137
7	21000	2930	0.91	634	7734	1700	58	140	258	154	118	170	92	81
8	9500	240	0.78	456	6552	503	73	63	21	132	85	144	27	102
9	9100	370	0.74	471	1855	2528	42	61	33	125	88	41	138	59
10	6600	800	0.52	325	4579	818	51	44	70	88	60	101	45	71
11	11800	610	0.87	364	5713	1178	80	79	54	147	68	125	64	112
12	26200	3600	0.41	585	4217	2012	84	175	317	69	109	93	109	117
13	11400	470	0.55	343	4061	2957	91	76	41	93	64	89	161	127
14	7200	1350	0.39	597	3242	665	73	48	119	66	111	71	36	102
15	38000	2470	0.68	1126	7658	1541	57	253	217	115	209	168	84	79
Average	15000	1137	0.59	538	4556	1838	72	100	100	100	100	100	100	100

factors which, in the analyst's view, should play a higher role in determining relative efficiencies, will have higher weight ranges than the (subjectively) less important ones. Similarly, where there is reason to believe that the effect of certain factors may be different in diverse situations, or when data sources are less reliable, the allowed spread of weights will be larger than in other cases.

A somewhat different approach[3] can be taken, remembering that all weights in the efficiency ratio can be expressed in terms of one of them, without changing the result. Thus, the bounding of factor weights can be specified, instead of absolute constraints. $OL_r \leqslant M_{rj} \leqslant OU_r$, in the form of ratios to the weight of one factor (say, M_{1j}) such as: $\underline{\pi}_r \cdot M_{1j} \leqslant M_{rj} \leqslant \bar{\pi}_r \cdot M_{1j}$ and $\underline{\omega}_i \cdot M_{1j} \leqslant N_{ij} \leqslant \bar{\omega}_i \cdot M_{1j}$. This approach is illustrated in Fig. 2. Again, the multipliers $\underline{\pi}_r$, $\bar{\pi}_r$, $\underline{\omega}_i$ and $\bar{\omega}_i$, indicate the importance assigned by the analysts to the different factors, relative to one of them (the first output in this case). The differences within each pair $\underline{\pi}_r$, $\bar{\pi}_r$, as well as $\underline{\omega}_i$ and $\bar{\omega}_i$, specify the ranges within which each factor weight is allowed to vary.

All modes of imposing bounds on factor weights introduce additional information into DEA efficiency analyses. This information represents the attitudes of the analysts, and their views on the relative importance of the dimensions along which efficiency is measured. Consequently, screening of candidate factors, while deciding on the ones to enter the analysis, should be a much more attentive process. Rather than taking a mere binary decision of whether to select or discard a factor, candidate outputs and inputs can, and sometimes should, be rated by their importance to the desired efficiency measure. After going through the necessary screening steps (see, e.g. [9]), bounds on the weights of remaining factors should be set to represent this relative importance. This kind of subjective intervention in the process of obtaining factor weights is also important in overcoming the problem of the non-uniqueness of the weights. It is well known that due to the structure of the linear programs constructed by DEA, where the number of DMUs (rows) is at least twice that of the number of factors (columns), a large number of alternative solutions (different factor weight vectors) may arise. Most DEA literature to date ignores this aspect and, in many applications of DEA, the first optimal weight vector generated by the software is used without looking for alternative solutions. By adding constraints (e.g. imposing bounds) we reduce the number of alternative solutions, but, we cannot guarantee uniqueness. However, by optimizing on one weight at a time the uniqueness problem is eliminated.

The process of determining bounds on factor weights is highly case-dependent, and no general rules can be laid down for it. A sample of techniques is, however, presented in the next section as a general guideline for possible approaches.

[3]The AR principle is a representative of this "relative" approach.

3. IMPOSING BOUNDS ON FACTOR WEIGHTS

In this section we present a short survey of models aimed at controlling factor weights in

DEA applications. The models are ordered from the least restrictive to the most restrictive approach. The choice of a specific technique depends mainly on the amount and kind of additional information which is desired to enter into the analysis.

3.1 The CCR model

The basic CCR model [3] (see Appendix) imposes no constraints on factor weights. Thus, it yields the highest relative ratings which are possible within the DEA framework. In presenting the original DEA model, the authors noted that allowing maximal flexibility in the choice of factor weights when evaluating each DMU (i.e. applying mere non-negativity constraints on M_{rj} and N_{ij}), may distort the analysis. Hence, they have introduced into the model a requirement for strict positivity of the weights through the ϵ parameter. Such an analysis is helpful when no prior information or preferences exist, and as a basis for possible weight constraining steps.

Applying the CCR model without any additional bounds allows each DMU to receive the most favorable efficiency score possible for it. However, this is accompanied with weight values which are sometimes unacceptable in real applications. In particular, obtaining an ϵ value for some weights implies that, in fact, the relevant factors do not influence the efficiency standing of that DMU. Also, cases in which the same factor receives drastically different weight values across the DMUs, may be managerially unacceptable.

3.2 General restriction of weight variation

When no information is available on the relative importance of the different factors, the amount of variation of weights within each factor can nevertheless be controlled. A possible technique is as follows:

(a) Run an unbounded CCR model, compile a "weight matrix" (see, for example, Table 1) and find the average weights M_r and N_i given to each factor, across all DMUs.

(b) Determine the amount of allowable variation within a weight for the same factor. For example, let the ratio of the highest value to the lowest one be $d:1$.

(c) Extend the basic CCR model by adding a set of bounding constraints of the type:

$$\frac{2 \cdot M_r}{1+d} \leqslant M_{rj} \leqslant \frac{2 \cdot d \cdot M_r}{1+d} \tag{2}$$

for each weight M_{rj} (when evaluating DMU$_j$). Apply similar constraints on the input weights.

(d) Run the "bounded" model.

A possible variation on this technique is to first truncate each vector of weights, by cutting off a certain percentage of extreme values on both sides, and find the average of the remaining values only.

It should be pointed out that, as long as absolute bounds are imposed, a transformation factor (fractional to linear formulation of DEA) should be added, resulting in a model of the type presented in [10].

3.3 Judgmental restriction of weight variation

Here, again, the bounds are largely determined on the basis of a "weight matrix" of unbounded runs. However, not all weights are treated equally. The analyst's attitude towards the different factors is brought forward by adopting central values M_r above or below the arithmetic average, as well as deciding on factor-specific ratios d_r, see e.g. [6].

3.4 Advance setting of bounds

An *a priori* determination of bounds on factor weights implies a strong initial position on the relative importance of factors and allowed spread of weights. Indeed, it is a declaration of policy on the merits by which the relative efficiency of DMUs is measured. This approach, as well as other weight bounding techniques, may result in infeasible situations. In that case, bounds should be gradually relaxed (either collectively as in 3.2, or by applying judgement as in 3.3) until feasible outcomes are obtained. Feasibility can be tested by running a formulation similar to (5) below.

3.5 Ratios among weights

Another approach to factor weight control is the setting of ratios within which weights are allowed to vary (see [12]). A typical set of constraints of this kind is

$$\underline{\alpha}_i \cdot N_1 \leqslant N_i \leqslant \bar{\alpha}_i \cdot N_1, \quad \forall i$$
$$\underline{\beta}_r \cdot M_1 \leqslant M_r \leqslant \bar{\beta}_r \cdot M_1, \quad \forall r. \qquad (3)$$

Note that the ratios $\bar{\alpha}_i/\underline{\alpha}_i$ (as well as $\bar{\beta}_r/\underline{\beta}_r$) serve as the d parameter in 3.2.

This formulation does not specify the numerical order of magnitude of the set of weights. However, after solving the DEA model, the entire set may be multiplied by a desired factor.

A stricter constraint of the same kind is obtained by tying the bounds on input factor weights to those of output weights, i.e. $\omega_i \cdot M_1 \leqslant N_i \leqslant \pi_i \cdot M_1$. Here M_1 serves as the numeraire for the entire set of bounds.

Controlling factor weights by means of ratios requires close attention to the units in which the respective factors are measured. This difficulty can be resolved by bringing all factor values to the same order of magnitude (see Section 2).

3.6 Miscellaneous constraints

In addition to individual bounds on particular weights, it may be desired to impose other relationships between factor weights. For example, suppose that 4 outputs are considered, and it is the analyst's view that variation in the first output should account for at least a half of the total output variation among DMUs. A possible constraint expressing this view would be: $M_1 > M_2 + M_3 + M_4$ (assuming all output values are first "normalized").

4. COMMON SET OF WEIGHTS

DEA assesses efficiencies by selecting for each DMU a particular set of input–output weights which offers the best possible value of its efficiency ratio. This reflects an important principle in DEA which allows for a flexible production function to be the instrument used to describe the relationship among inputs and outputs in the given system of DMUs. However, it might be useful in some cases to derive from the analysis a common set of weights (CSW) which can be applied to all the DMUs. Such a common set can serve as a yardstick to which the results of the ordinary ("flexible") DEA outcomes are compared. For example, consider three DMUs whose efficiency is determined in both ways as given below:

DMU	Unconstrained efficiency	CSW efficiency
1	0.77	0.75
2	0.73	0.4
3	0.75	0.68

Clearly, in the unconstrained analysis each of the three DMUs takes advantage of the flexibility in choosing the most appropriate weights and all three reach similar efficiency levels. However, applying the CSW concept results in a considerably lower rating for DMU_2. It follows that the rating for this DMU is highly sensitive to the selection of factor weights. In extreme cases, this may suggest a need to examine whether the operating conditions of the sensitive DMU are indeed similar to those of the rest of the analyzed group. This concept may lead to new kind of sensitivity analysis in DEA, one which will point out the more robustly efficient (and nearly-efficient) DMUs.

The notation of CSW was first introduced in [10], and is further developed here. As in the case of setting bounds on factor weights, deriving a CSW requires additional (often subjective) information to be introduced into the analysis. Following is a sample of possible approaches.

4.1 Central values between bounds

In the absence of other information, looking for central values for all the weights is a straightforward approach to generate a CSW. Starting from a bounded model, this can be achieved by expressing the deviation from either bound as a fraction of the range between the upper and lower bounds. Assuming the same deviation from bounds across all DMUs, we get:

$$\begin{aligned}
&\text{Max } \phi \\
&\text{s.t.} \\
&\sum_{r=1}^{R} M_r \cdot Y_{rj} - \sum_{i=1}^{I} N_i \cdot X_{ij} \leqslant 0, \quad \forall j \\
&\text{OL}_r + \phi \cdot (\text{OU}_r - \text{OL}_r) \leqslant M_r \leqslant \text{OU}_r \\
&\quad - \phi \cdot (\text{OU}_r - \text{OL}_r), \quad \forall r \\
&\text{IL}_i + \phi \cdot (\text{IU}_i - \text{IL}_i) \leqslant N_i \leqslant \text{IU}_i - \phi \cdot (\text{IU}_i - \text{IL}_i), \quad \forall i.
\end{aligned} \qquad (4)$$

In cases were $\phi = 0.5$ (i.e. all weight values are placed at the middle of their respective ranges), and none of the DMUs is fully efficient, all output weights can be increased (and/or input weights decreased) by a minimal proportion until an efficient DMU is reached.

Table 2. The unbounded weight matrix

j	M_1	M_2	M_3	N_1	N_2	N_3	N_4	Efficiency
1	42.23	*0.00	4063.68	63.46	21.39	*0.00	*0.00	1.000
2	13.89	*0.00	*11259.82	*0.00	*0.00	*0.00	*229.24	0.900
3	55.56	0.00	*0.00	*0.00	22.86	15.54	21.59	1.000
4	56.17	0.00	8929.40	*199.62	5.47	0.00	52.29	1.000
5	*92.59	0.00	*0.00	*302.82	15.52	0.00	*0.00	1.000
6	36.29	0.00	2410.96	109.10	*0.00	20.47	0.00	0.741
7	*0.00	*34.13	0.00	0.00	0.00	*56.21	14.03	1.000
8	15.14	13.04	10574.70	6.38	11.52	43.03	0.00	1.000
9	*69.46	0.00	4971.83	33.61	*28.34	0.00	*75.24	1.000
10	*0.00	29.87	*12458.02	167.28	0.00	55.79	0.00	0.887
11	0.00	19.89	10099.37	99.64	9.77	6.76	0.00	1.000
12	0.00	27.78	0.00	0.00	*23.71	0.00	0.00	1.000
13	46.17	0.00	6506.39	194.02	8.24	0.00	0.00	0.884
14	0.00	*74.07	0.00	0.00	6.42	*119.09	0.00	1.000
15	26.32	0.00	0.00	0.00	10.83	7.31	10.23	1.000
Average	30.25	13.25	4751.61	78.40	10.94	21.61	26.84	
Average	26.52	8.23	4323.30	61.23	10.18	13.54	8.92	
U	39.79	12.35	6484.95	91.84	15.28	20.30	13.38	1.50 (Av.)
L	13.26	4.12	2161.65	30.61	5.09	6.77	4.46	0.50 (Av.)
U	35.36	10.98	5762.96	81.61	13.57	18.04	11.89	1.33 (Av.)
L	17.69	5.49	2883.64	40.84	6.79	9.03	5.95	0.67 (Av.)
								Truncated (less asterisked)

4.2 A preferred order of factors

This approach starts by arranging the various factors in a descending order of importance. The "importance" in this context, can be expressed either in terms of associating more discriminating power to a particular factor by according it a large weight, or conversely, when having less confidence in the significance of a certain factor, by minimizing its weight value. Starting with the bounded model and following the preferred order, each factor is pushed to its bound (upper for the more important factors, lower for the less desired ones). This is achieved by solving a model of the following kind (illustrated below for the case of a favorable factor $M_{r'}$).

$$\text{Max } M_{r'}$$

$$\text{s.t.}$$

$$\sum_{r=1}^{R} M_r \cdot Y_{rj} - \sum_{i=1}^{I} N_i \cdot X_{ij} \leqslant 0, \quad \forall j$$

$$\mathrm{IL}_i \leqslant N_i \leqslant \mathrm{IU}_i, \quad \mathrm{OL}_r \leqslant M_r \leqslant \mathrm{OU}_r, \quad \forall i, r. \qquad (5)$$

When the outcome of the model is that the particular bound was reached, the corresponding weight is fixed at the bound and the same procedure is applied to the next factor. This process continues until, at some stage, the bound is not reached. This signifies that there is no more "slack" in the system, and the resultant set of weights is taken as a CSW. Such a CSW will show at least one DMU as fully efficient, with the efficiencies of the other DMU measured relative to the efficient ones. A similar approach was first mentioned in [6], and demonstrated in [10].

4.3 Maximizing the average efficiency of all DMUs

One of DEA's unique features is the attempt to accord each individual DMU the best possible efficiency score. When a CSW is sought, management may adopt a similar line of thought in looking for the set of weights which will maximize the overall (or the average) efficiency standing over the entire set of DMUs. The model proposed here (see Appendix), is formulated as a non-linear programming problem which finds the CSW that maximizes the average efficiency while adhering to the basic envelopment constraints.

4.4 Maximizing the number of efficient DMUs

This is a similar approach to the one presented above, but here the focus is only on the fully efficient DMUs. Hence, the approach is to find a CSW which maximizes the number of efficient DMUs even if it decreases the average efficiency standing across all the DMUs. The corresponding model (see Appendix) requires a set of binary variables (denoted as z_js) to indicate whether a unit is efficient or not. However, using known approximation techniques, one is able to reduce this mixed integer programming problem into an ordinary non-linear program.

Table 3. Summary of results

Data		Raw								"Normalized"			
Bounding technique		Unbounded		d-ratio 3:1		d-ratio 2:1		Prescribed		$R\{M_1, N_1\}$		$R\{M_1\}$	
	Run	A		B		C		D		E		F	
		U	L	U	L	U	L	U	L	U	L	U	L
Bounds on factor heights	M_1	—	—	39.79	13.26	35.36	17.69	80	20	—	—	—	—
	M_2	—	—	12.75	4.12	10.98	5.49	40	10	$1.0M_1$	$0.2\ M_1$	$1.0M_1$	$0.2M_1$
	M_3	—	—	6484.95	2161.65	5762.96	2883.64	30	10	$0.5M_1$	$0.1\ M_1$	$0.5M_1$	$0.1M_1$
	N_1	—	—	91.84	30.61	81.61	40.84	60	10	—	—	$2.0M_1$	$0.5M_1$
	N_2	—	—	15.28	5.09	13.57	6.79	80	10	$4.0N_1$	$0.25N_1$	$2.0M_1$	$0.5M_1$
	N_3	—	—	20.30	6.77	18.04	9.03	80	20	$4.0N_1$	$0.25N_1$	$2.0M_1$	$0.5M_1$
	N_4	—	—	13.38	4.46	11.89	5.95	20	2	$0.4N_1$	$0.10N_1$	$0.2M_1$	$0.02M_1$
Relative efficiencies	$j = 1$	1.000		1.000		1.000		0.955		0.852		0.828	
	2	0.900		0.563		0.541		0.525		0.477		0.437	
	3	1.000		0.883		0.757		0.857		0.904		0.752	
	4	1.000		0.894		0.802		0.881		0.753		0.727	
	5	1.000		0.742		0.662		0.732		0.737		0.753	
	6	0.741		0.696		0.661		0.662		0.638		0.627	
	7	1.000		1.000		0.976		1.000		0.991		0.931	
	8	1.000		0.957		0.839		1.000		0.936		0.787	
	9	1.000		1.000		0.932		0.874		0.853		0.702	
	10	0.887		0.816		0.758		0.812		0.705		0.654	
	11	1.000		1.000		0.953		1.000		0.881		0.819	
	12	1.000		1.000		1.000		1.000		1.000		1.000	
	13	0.884		0.727		0.670		0.631		0.589		0.576	
	14	1.000		0.798		0.704		0.920		0.779		0.718	
	15	1.000		1.000		1.000		1.000		1.000		1.000	
Average		0.961		0.872		0.817		0.857		0.806		0.754	
Num. fully eff.		11		6		3		5		2		2	

5. NUMERICAL EXAMPLE

Assume that in analyzing a group of 15 DMUs, all carrying out the same tasks and utilizing similar outputs, 3 outputs and 4 inputs were chosen for assessing relative efficiencies. The raw data gathered, and the respective values when normalized to an average of 100 in each factor, are given in Table 1. Running the basic CCR model yields the results shown in the

Table 4. CSW efficiencies

				Max. No. of eff. DMUs		Max. average effciency	
	Data	"Normalized"		see @	as in D	see @	as in D
	Run	G	H	K	L	M	N
Common set of weights	M_1	45.26	80.00		26.300		20.000
	M_2	22.63	26.01		11.327		10.000
	M_3	18.42	10.00		17.080		30.000
	N_1	38.95	60.00		24.234		11.965
	N_2	50.53	80.00		10.000		25.655
	N_3	54.74	80.00		39.884		44.183
	N_4	12.42	2.00		12.417		2.000
Relative efficiencies	$j = 1$	0.579	0.546	1.000	0.665	1.000	0.890
	2	0.305	0.281	0.553	0.356	0.632	0.409
	3	0.462	0.483	0.557	0.421	0.539	0.506
	4	0.419	0.376	0.804	0.538	0.985	0.604
	5	0.485	0.531	0.608	0.471	0.739	0.447
	6	0.486	0.497	0.697	0.559	0.681	0.609
	7	0.781	0.690	1.000	1.000	1.000	1.000
	8	0.432	0.364	0.846	0.710	0.751	0.889
	9	0.421	0.354	0.793	0.469	0.924	0.637
	10	0.482	0.403	0.756	0.672	0.740	0.776
	11	0.542	0.473	1.000	0.760	1.000	0.923
	12	1.000	1.000	1.000	1.000	1.000	1.000
	13	0.350	0.335	0.637	0.388	0.801	0.453
	14	0.545	0.492	0.561	0.612	0.463	0.835
	15	0.832	0.823	1.000	1.000	0.879	1.000
Average efficiency		0.541	0.510	0.787	0.641	0.809	0.732
Num. fully eff.		1	1	5	3	4	3

@ OU = 10* Av $\{M_r\}$, IU = 10* Av $\{N_i\}$
@ OL = 0.1* Av $\{M_r\}$, IL = 0.1* Av $\{N_i\}$

upper part of Table 2. Note that out of 105 weight values, 46 are virtual zeros. Also, 11 out of the 15 DMUs appear to be fully efficient—when compared to the others.

A first bounding attempt was made, following the procedure outlined in 3.2. The two extreme entries on both sides, in each column of the weight matrix, were cut off, and an average of the remaining 11 central values calculated. Choosing a ratio of 3:1 between the highest allowed weight and the lowest one for each factor, appropriate bounds [using expression (2)] were determined. A correspondingly bounded model was then run. The same procedure was repeated for a ratio of 2:1 (see the bottom part of Table 2). Efficiencies obtained this way are listed in columns B and C in Table 3. The number of "fully" efficient DMUs dropped from 11 to 6 then to 3, and the average efficiency went down from 0.961 in the unbounded version to 0.817 in the 2:1 bounded run.

The next bounding attempt involved a decision on the relative importance of the 7 factors and the ranges within which they may be allowed to vary. After normalizing the numerical values of all the factors (as in Table 1), the following set of bounds was placed on the factor weights. (These bounds represent subjective information which an analyst may wish to include in the analysis.)

Factor weight	M_1	M_2	M_3	N_1	N_2	N_3	N_4
Upper bounds (OU, IU)	80	40	30	60	80	80	20
Lower bounds (OL, IL)	20	10	10	10	10	20	2

In this example, the first output (y_1) and the third input (x_3) are assigned a relatively high possible weight with a d-ratio of 4:1, while the fourth input (x_4) is given a low upper bound as well as a ratio of 10:1. The corresponding efficiencies are listed in column D of Table 3. As expected, the relative efficiencies, taking into account the analyst's attitude towards the various factors, may be considerably different than the outcomes reached without such prior considerations.

Columns E and F in Table 3 list relative efficiencies obtained with the models of the type presented in 3.5. In one case (column E), the ratio between input weights and output weights is left unconstrained, in the other case (column F) all factors are interrelated.

The results of several attempts to locate meaningful CSWs are presented in Table 4. Column G lists CSW efficiencies when model (4) is applied. The prescribed bounds (column D, Table 3) were used as a starting point and central values between these were obtained (the ϕ value reached in this case was 0.421). DMU_{12} turned out to be the one that hit the bounds first. Hence, it is considered fully efficient and the efficiencies of all other DMUs are calculated relative to the CSW which rendered it the efficient position. As expected, the efficiency figures here are significantly lower than those achieved with the previous (more flexible) DEA models. However, the magnitude of the drop in relative efficiency is not uniform, a fact which may suggest further analysis of the operating conditions of those DMUs which suffered the largest decreases in their evaluation.

Another attempt at finding a CSW was made according to the procedure suggested in 4.2. Again, the starting points were the prescribed bounds (column D, Table 3). Ordering the factors by their assumed importance, the procedure starts by attempting to accord y_1 the maximum possible weight (within the accepted bounds). Model (5) was run, whereby M_1 reached its upper bound. Fixing M_1 at this level, the procedure continued. Next, N_4 was minimized (since x_4 was considered the least important among all the factors). Again, N_4 reached its bound. At the third step, M_2 fell short of its upper bound ($M_2 = 26.01$) and the procedure reached an end. The resultant CSW is shown at the upper part of column H. It is interesting to note that in this case the relative efficiencies which correspond to these two rather unrelated CSW approaches (G and H) are not drastically different from one another.

The CSW models which maximize the average efficiency and the number of efficient DMUs were run with the additional constraints which were given as optional in the Appendix. The results are shown in columns K through N in Table 4. The optional constraints were applied in two ways. First, a general restriction was imposed for each output weight to be within 0.1–10 times of the average weight across all outputs, and similarly for the inputs (columns K and M). Then, another run was done, which applied to bounds prescribed in column D (columns L and N). In both cases, the former (less restrictive) run produced larger efficiency averages and more fully efficient DMUs. The trade-off between the two approaches represented in the two models is exemplified by observing the last two lines in colunns K and M. The first model, with the objective to maximize average efficiency, indeed resulted in a gain of $0.809 - 0.787 = 0.022$ in the average efficiency, but, this was achieved at the cost of losing one fully efficient DMU.

6. DISCUSSION

This paper develops and presents a number of alternative methods to limit the range within which factor weights are allowed to vary in DEA. All of these methods involve additional information which is entered into the analysis in the form of constraints, bounds or different objective functions. A natural criticism of such an approach may be that in doing so, the objectivity of DEA is lost. However, one should bear in mind the context in which DEA is most relevant. When an organization is capable of describing all its inputs, goals and outputs in pure economic terms (i.e. market prices, costs, etc.), one does not need DEA, and standard economic or engineering approaches are appropriate. However, DEA is usually applied to much more complex situations in which many non-commensurate factors, some qualitative in nature, participate in the performance evaluation of the DMUs. In choosing the factors to enter the analysis, one has already expressed strong opinion on the importance of these factors versus others which were left out. Imposing bounds on factor weights renders this process more flexible by defining the importance of the different factors. In other words, it treats the "importance variable" as a real variable rather than a binary one. Thus, factors are prevented from gaining excessive weight values, or from being virtually neglected.

Obviously, the tighter the control on factor weights, i.e. the more stringent the requirements, the less efficient the DMUs seem to be. It is, of course, for the analysts to decide on the severity of the constraints to be imposed in each case. When the bounding approach is pushed to its extreme, a CSW can be found. This may be of value in some cases, for the analysis of gaps between the respective efficiency scores and frontier performance.

Different bounding techniques may render significantly different results. The apparent problem, in controlling factor weights, lies, therefore, in the question of how to choose an appropriate set of bounds. This requires that prior to conducting the actual efficiency assessment, the analysts will have a clear purpose in mind, will have a general notion of the relative importance of the various goals and an idea about weights can be associated with the inputs. This is not an unreasonable requirement. Moreover, it may be of interest to observe how efficiencies of DMUs vary, assuming different points of view and with different attitudes towards the relative importance of the pertinent factors. Thus, control of factor weights extends the applicability of DEA and adds new dimensions for possible anaysis of efficiency outcomes.

APPENDIX

Notation

j, k = indices of DMUs, $j, k = 1, \ldots, J$

r = index of outputs, $r = 1, \ldots, R$

i = index of inputs, $i = 1, \ldots, I$

$e_j h_k$ = efficiency score for DMU_k

Y_{rj} = the value of the rth output of the jth DMU

X_{ij} = the value of the ith output of the jth DMU

M_{rj} = factor weight for output r of the jth DMU

$\underline{\alpha}_i, \bar{\alpha}_i, \underline{\beta}_r, \bar{\beta}_r, \underline{\pi}_r, \bar{\pi}_r, \underline{\omega}_i, \bar{\omega}_i$ = various multipliers applied to the factor weights

N_{ij} = factor weight for input i of the jth DMU

OL_r, OU_r = lower and upper bounds (respectively) on output factor r

IL_i, IU_i = lower and upper bounds (respectively) on input factor i

z_j = a 0–1 indicator of efficient units

ϵ = a non-Archimedean infinitesimal

d = ratio between highest to lowest weight values

The CCR ratio model

For each DMU_k solve:

$$\text{Max } h_k = \frac{\sum_{r=1}^{R} M_{rk} \cdot Y_{rk}}{\sum_{i=1}^{I} N_{ik} \cdot X_{ik}}$$

s.t.

$$\frac{\sum_{r=1}^{R} M_{rk}\cdot Y_{rj}}{\sum_{i=1}^{I} N_{ik}\cdot X_{ij}} \leqslant 1, \quad \forall j$$

$$\frac{M_{rk}}{\sum_{i=1}^{I} N_{ik}\cdot X_{ij}} \geqslant \epsilon, \quad \forall r$$

$$\frac{N_{ik}}{\sum_{i=1}^{I} N_{ik}\cdot X_{ij}} \geqslant \epsilon, \quad \forall i.$$

The Cone Ratio Model

For each DMU_k solve:

$$\text{Max } h_k = \frac{\sum_{r=1}^{R} M_{rk}\cdot Y_{rk}}{\sum_{i=1}^{I} N_{ik}\cdot x_{ik}}$$

s.t.

$$\sum_{j=1}^{J}\left[\sum_{i=1}^{I} N_{ik}\cdot X_{ij} - \sum_{r=1}^{R} M_{rk}\cdot Y_{rj}\right] \in U$$

$$M_{rk} \in V, \quad \forall r; N_{ik} \in W, \quad \forall i, \quad (V, W \neq \emptyset)$$

where V, W and U are closed convex cones with non-empty interior defined by:

$$V \subset E_+^R, \quad W \subset E_+^I, \quad U \subset E^J$$

$$\delta_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in -U^*, \quad \forall j$$

$$U^* = \{u | u' \cdot u \leqslant 0, \forall u' \in U\}, \quad \text{the "polar cone" of } U.$$

In the special case where

$$V = E_+^R, \quad W = E_+^I, \quad U = E_+^J$$

the cone ratio model is reduced to the original CCR model.

The AR Principle

Use one of the input weights (say N_1) as the yardstick for all other input weights and similarly, M_1 with respect to all the outputs. An AR may be specified by $(R + I - 2)$ homogeneous linear inequalities for separable cones (the DMU's index is omitted for convenience):

$$\underline{\alpha}_i \cdot N_1 \leqslant N_i \leqslant \bar{\alpha}_i \cdot N_1, \quad \forall i \text{ (input cone)}$$

$$\underline{\beta}_r \cdot M_1 \leqslant M_r \leqslant \bar{\beta}_r \cdot M_1, \quad \forall r \text{ (output cone)}$$

where $\underline{\alpha}_i$, $\bar{\alpha}_i$, $\underline{\beta}_r$, $\bar{\beta}_r$ are non-negative scalars estimated on the basis of observations and expert opinion.

A CSE Model to Maximize the Average Efficiency of all DMUs

$$\text{Max} \frac{1}{n} \cdot \sum_{j=1}^{J} e_j$$

s.t.

$$e_j = \frac{\sum_{r=1}^{R} M_r \cdot Y_{rj}}{\sum_{i=1}^{I} N_i \cdot X_{ij}}, \quad \forall j$$

$$0 \leqslant e_j \leqslant 1, \quad \forall j$$

$$N_i \geqslant \epsilon, \quad M_r \geqslant 0, \quad \forall i, r$$

optional:

$$\alpha_1 \cdot \left(\frac{1}{I}\sum_{i=1}^{I} N_i\right) \leqslant N_i \leqslant \alpha_2 \cdot \left(\frac{1}{I}\sum_{i=1}^{I} N_i\right), \quad \forall i$$

$$\beta_1 \cdot \left(\frac{1}{R}\sum_{r=1}^{R} M_r\right) \leqslant M_r \leqslant \beta_2 \cdot \left(\frac{1}{R}\sum_{r=1}^{R} M_r\right), \quad \forall r.$$

A CSW Model to Maximize the Number of Efficient DMUs

$$\text{Max} \sum_{j=1}^{J} z_j$$

s.t.

$$e_j = \frac{\sum_{r=1}^{R} M_r \cdot Y_{rj}}{\sum_{i=1}^{I} N_i \cdot X_{ij}}, \quad \forall j$$

$$e_j - z_j \geqslant 0, \quad \forall j$$

$$0 \leqslant e_j \leqslant 1, \quad \forall j$$

$$z_j \in \{0, 1\}, \quad \forall j$$

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ADDRESS FOR CORRESPONDENCE: *Dr Boaz Golany, Faculty of Industrial Engineering and Management, Technion–Israel Institute of Technology, Haifa 32000, Israel.*