

# Robust Multi Echelon Multi Period Inventory Control

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We consider the problem of minimizing the overall cost of a supply chain, over a possibly long horizon, under demand uncertainty which is known only crudely. Under such circumstances, the method of choice is Robust Optimization, in particular the Affinely Adjustable Robust Counterpart (AARC) method which leads to tractable deterministic optimization problems. The latter is due to a recent re-parametrization technique for discrete time linear control systems. In this paper we model, analyze and test an extension of the AARC method known as the Globalized Robust Counterpart (GRC) in order to control inventories in serial supply chains. A simulation study demonstrates the merit of the methods employed here, in particular, it shows that a good control law that minimizes cost achieves simultaneously good control of the bullwhip effect.

**KEYWORDS** Multi Echelon Supply Chains, Multi-Period Inventory Control, Robust Optimization, Affinely Adjustable Robust Optimization, Globalized Robust Counterpart, Bullwhip Effect

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## 1. Introduction.

A Supply chain is a network of nodes, arranged in echelons that correspond to their relative position in the network. The nodes are interconnected through supply-demand relationships. The dynamic state of such chains can be described through the inventory levels at each node at any given time. These chains serve external demand which generates orders to the most downstream echelon and are served by external supply which responds to the orders of the most upstream echelon. The problem of multi-echelon multi-period inventory control has been investigated as early as the 1950's by researchers such as Arrow et al. (1958) and Whitin (1957) and later by Love (1979) and Forrester (1973). The main challenge in these problems is in controlling the inventory levels by determining the size of the orders for each echelon in each time period so as to optimize a given objective function. The source of difficulty in these problems lies in the inherent uncertainty that characterizes the demand processes.

Solving this type of uncertain control problems with classical methods such as dynamic programming (DP) and Stochastic programming (SPR) is not realistic as the dimensions of the problem grow exponentially with the number of decisions, states and periods. Fortunately, the problem's characteristics enable us to formulate it as a general control problem which can be solved with a linear controller via the Robust Optimization (RO) methodology developed by Ben-Tal and Nemirovski (2002).

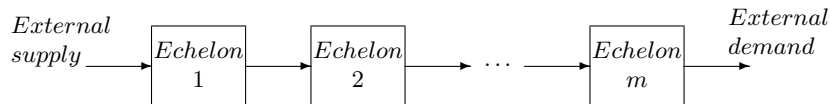
The paper is organized as follows. In the next section we present the problem, review some of the approaches that were proposed in order to solve it and discuss the nature of its uncertain data. Later, we reformulate the problem as a general control problem and construct an appropriate robust optimization model to solve it. Then, we present the results of our control approach while focusing on the reduction of the bullwhip effects that typically characterize such systems. We study the performance of the model we developed under different settings and carry out various sensitivity analysis.

## 2. Problem Description.

### 2.1. Problem Formulation and History.

Controlling inventory levels in supply chains is done by determining for each node in the chain what to order, at what time and in what quantity. There are many kinds of supply chains, including: *distribution* (or arborescence) chains, where each node supplies one or several downstream nodes; *assembly* (or coalescence) chains, where each node is supplied by one or several upstream nodes; and *serial* chains, where each node supplies only one downstream node and is supplied by only one upstream node and where only the node in the last echelon has external demand and only the node in the first echelon may have an external supplier. Although serial chains may seem too trivial to explore, they are actually very important, because their simple structure is the building block of any and all supply chains. For example, one can model a distribution chain as a serial chain with aggregated demands (see, e.g., the study by Cachon et al. (2005)) to compensate for the lack of detailed information. Thus, investigating serial chains can yield important insight on the behavior of supply chains in general. Indeed, the literature on supply chains contains many articles that analyze the performance of serial chains. Early models of this kind were developed by Arrow et al. (1958), Whitin (1957), Love (1979) and Forrester (1973). But, as the computational power in these days was quite limited, these initial attempts were, for the most part, unsuccessful even for simple serial chains. Consequently, there were virtually no further attempts made for most of the 1970's-1980's. A renewed wave of interest in arose in the 1990's and other models dealing with serial chains started to emerge. Some examples of such models can be found in Lin et al. (2004), Zhang (2005) and Saab and Corrêa (2005) which base their model on a model which is described in Kirkwood (1998). From now on we will focus our discussion on the structure which is depicted in figure 1.

**Figure 1** A serial supply chain



We treat a serial chain as a network that is a discrete time dynamical system. Let us denote by  $j = 1, 2, \dots, m$ , the index of the echelons, where echelon  $j$  is echelon's  $j + 1$  predecessor for  $j = 1, 2, \dots, m - 1$ . There is an external demand  $d_t$  faced by echelon  $m$  for  $t = 1, 2, \dots, n$ , where  $n$  is the number of time periods constituting the planning horizon.

Let us denote the amount echelon  $j$  orders from echelon  $j + 1$  at the beginning of period  $t$  by  $x_t^j \geq 0$  and the inventory level in echelon  $j$  at the end of time period  $t$  by  $y_t^j$ . The initial inventory level in echelon  $j$  is denoted by  $z^j$ . We assume that there is a one-to-one correspondence between the units that must be supplied and the demand in each of the echelons. We also assume that there exist some non-negative delays between the time that an order is placed and the time it is supplied. There are 3 types of delays: (1) *information delay*: the time it takes the information about the order to reach the preceding echelon, (2) *manufacturing delay*: the time it takes to manufacture or assemble the order (measured from the time the order is received), and (3) *lead time*: the time it takes the order to travel from its origin to its destination. We will denote the 3 delays in each echelon  $j$  by  $I(j)$ ,  $M(j)$  and  $L(j)$ , respectively, and assume that all 3 are non-negative integers<sup>1</sup>.

<sup>1</sup> These parameters should be integers because of the discrete time interval used.

$I(m+1)$  denotes the information delay between the external demand and echelon  $m$ . Consequently, the relationship that must be satisfied between the echelons is:

$$\begin{aligned} y_t^j &= y_{t-1}^j + x_{t-(I(j)+M(j-1)+L(j))}^j - x_{t-(I(j+1)+M(j))}^{j+1} & \forall j \in \{1, \dots, m-1\} \\ y_0^j &= z^j & \forall j \in \{1, \dots, m\} \\ y_t^m &= y_{t-1}^m + x_{t-(I(m)+M(m-1)+L(m))}^m - d_{t-(I(m+1)+M(m))} & \end{aligned} \quad (1)$$

which is simply a mathematical way of saying that the change in inventory level from one period to the next is equal to the quantities received minus the requirements. Negative levels of inventory, which may occur, represent unsatisfied requirements or *backlogging*. Notice however that although the requirements are not satisfied they reach the successive echelon in time, this can be explained by a “borrowing” strategy — when a certain echelon cannot satisfy an order it goes to “the store next door”, that can supply the same item, and borrows the quantity needed to satisfy the order, which it will return on a later date. Such a loan is obviously accompanied by a cost; we will discuss this cost later. A necessary assumption for this kind of strategy to be possible is that there is always enough on-hand stock of the item, somewhere among the external suppliers and that it is available for borrowing. A simpler version of this model is described by Love (1979).

We formulate our optimization problem with the most straightforward objective function — minimization of the total cost, which is composed of three types of costs: (1) buying or manufacturing costs, (2) inventory holding costs, and (3) backlogging (or borrowing) cost. We will denote the buying\manufacturing cost per item at echelon  $j$  and time period  $t$  by  $c_t^j$ , the holding cost per item per unit of time in echelon  $j$  and time period  $t$  by  $h_t^j$  and the backlogging (or shortage) cost per item per unit of time in echelon  $j$  and time period  $t$  by  $p_t^j$ . The index  $t$  of the various costs allows us to consider capitalization (for instance  $c_t^j = c^j(1+r)^{t-1}$ ), which can greatly impact the cost when the planning horizon is long.

Instead of minimizing the cost elements above, one may opt to control supply chains by minimizing or even eliminating the “bullwhip effect” — the amplification of demand variability from a downstream site to an upstream site (see Lee et al. (1997b)). Reducing this effect has implications beyond cost minimization since bullwhip peaks and ebbs often cause disruptions that are difficult to quantify (e.g., loss of reputation and goodwill among customers and suppliers). Bullwhip effects may be caused by the use of heuristics (Love (1979) and Forrester (1973)); by irrational behavior of the supply chain members, as illustrated in the “Beer Distribution Game” in Sterman (1989), or as a result of the strategic interactions among rational supply chain members in Lee et al. (1997a). There are many real-world evidences of the occurrence of the bullwhip effect. Examples include diapers (Lee et al. (2004)), TV-sets (Holt et al. (1968)), food products (Lee et al. (1997a), Hammond (1994)) pharmaceutical products (Cachon et al. (2005)), and more. Terwiesch et al. (2005) note that the semiconductor equipment industry is more volatile than the personal computer industry and Blanchard (1983) shows evidence of bullwhip existence through an empirical study he conducted in the automotive industry.

There were many attempts to construct strategies aimed at minimizing the bullwhip effect. This include methodologies such as the z-transform in Lin et al. (2004) and Disney et al. (2004) and numerical simulation conducted by Saab and Corrêa (2005) to compare three models suggested by Forrester (1973), Kirkwood (1998) and Sterman (2000). All these models are very specific and they do not provide a general or optimal solution. A more radical suggestion was given by Forrester, who showed that elimination of nodes (echelons) from the supply chain significantly helps to reduce the undesired fluctuation. This solution can not be widely adopted because of real-world constraints on the structure and operation of supply chains.

The objective of most of the studies we mentioned so far was to minimize the bullwhip effect, which is defined as either the ratio or difference between the order variance and the demand variance (Chen et al. (2000), Cachon et al. (2005), Zhang (2005)). Some try to minimize the inventory

variance, or some combination of these in Disney et al. (2004) or other stability performance measures as in Lin et al. (2004). In contrast, in this paper we will try to apply an economic rationale to our control problem, since we want to control the chain not merely for the sake of stabilizing the system for operational reasons but first and foremost in order to minimize cost. Lin et al. (2004) claim that trying to build a control system using costs is too dependent on the actual values assigned to these costs. Nevertheless, we believe that a general, yet parameter dependent, methodology can be developed and it will yield good control results no matter what the parameter values are.

We will use the predefined notation for inventory levels and ordered amounts. The system's dynamics is as described previously in equation (1). The optimization problem is therefore described by the linear program (2), where  $d_t \in D_t$ ,  $z^j \in Z^j$  denote the set of possible values of the data. In order to stabilize the system we may also want to incorporate constraints such as  $\underline{a}^j \leq y_t^j \leq \bar{a}^j$  and  $x_t^j \leq b^j$  for all  $t \in \{1, \dots, n\}$  and  $j \in \{1, \dots, m\}$ .

$$\begin{aligned}
& \min_{y,x} \sum_{j,t} [c_t^j x_t^j + \max(h^j y_t^j, -p^j y_t^j)] \\
& \text{s.t.} \\
& \left. \begin{aligned}
y_t^j &= y_{t-1}^j + x_{t-(I(j)+M(j-1)+L(j))}^j \\
&\quad - x_{t-(I(j+1)+M(j))}^{j+1} \quad \forall j \in \{1, \dots, m-1\} \\
y_t^m &= y_{t-1}^m + x_{t-(I(m)+M(m-1)+L(m))}^m - d_{t-(I(m+1)+M(m))} \\
x_t^j &\geq 0 \\
y_t^j &\geq 0 \\
y_0^j &= z^j
\end{aligned} \right\} \forall t \in \{1, \dots, n\} \quad (2)
\end{aligned}$$

Notice that this is not a Linear Programming (LP) problem. In order to transform the problem to a linear form, we need to substitute the *max* expression in the objective function. This is done by adding the auxiliary variables  $w_t$ . Furthermore, in order to simplify the formulation we will also denote  $T^L(j) = I(j) + M(j-1) + L(j)$ , the delay between the time an order is placed and received in echelon  $j$ , and  $T^M(j) = I(j+1) + M(j)$  the delay between the time an order is placed in echelon  $j+1$  and shipped from echelon  $j$ . Thus, we get the following formulation:

$$\begin{aligned}
& \min_{y,x} \sum_{j,t} [c_t^j x_t^j + w_t^j] \\
& \text{s.t.} \\
& \left. \begin{aligned}
y_t^j &= y_{t-1}^j + x_{t-T^L(j)}^j - x_{t-T^M(j)}^{j+1} \quad \forall j \in \{1, \dots, m-1\} \\
y_t^m &= y_{t-1}^m + x_{t-T^L(m)}^m - d_{t-T^M(m)} \\
w_t^j &\geq h_t^j y_t^j \\
w_t^j &\geq -p_t^j y_t^j \\
y_t^j &\geq \underline{a}^j \\
y_t^j &\leq \bar{a}^j \\
x_t^j &\leq b^j \\
x_t^j &\geq 0 \\
w_t^j &\geq 0 \\
y_0^j &= z^j
\end{aligned} \right\} \forall t \in \{1, \dots, n\} \quad (3)
\end{aligned}$$

The next step will be to eliminate the redundant equality constraints, by applying them recursively, and achieving the final formulation given by (4).

In reality, the demand and initial inventory values are unknown or uncertain. These uncertain parameters - the demand  $d = \{d_t\}_{t=1, \dots, n}$  and initial inventory values  $z = \{z^j\}_{j=1, \dots, m}$  belong to uncertainty sets so that  $d_t \in D_t$ ,  $z^j \in Z^j$  and the sets  $D = \{D_t\}_{t=1, \dots, n}$ ,  $Z = \{Z^j\}_{j=1, \dots, m}$  denote the

possible values of these parameters. Thus, formulation (4) in fact represents a family of LPs - one for each possible realization of the uncertain data.

$$\begin{aligned}
& \min_{\sigma, w, x} \sigma \\
& \text{s.t.} \\
& \left. \begin{aligned}
& \sigma \geq \sum_{j,t} [c_t^j x_t^j + w_t^j] \\
& w_t^j \geq h_t^j (z^j + \sum_{t'=1}^t (x_{t'-TL(j)}^j - x_{t'-TM(j)}^{j+1})) \\
& w_t^j \geq -p_t^j ((z^j + \sum_{t'=1}^t (x_{t'-TL(j)}^j - x_{t'-TM(j)}^{j+1})) \\
& \underline{a}^j \leq z^j + \sum_{t'=1}^t (x_{t'-TL(j)}^j - x_{t'-TM(j)}^{j+1}) \\
& \bar{a}^j \geq z^j + \sum_{t'=1}^t (x_{t'-TL(j)}^j - x_{t'-TM(j)}^{j+1}) \\
& w_t^m \geq h_t^m (z^m + \sum_{t'=1}^t (x_{t'-TL(m)}^m - d_{t'-TM(m)})) \\
& w_t^m \geq -p_t^m ((z^m + \sum_{t'=1}^t (x_{t'-TL(m)}^m - d_{t'-TM(m)})) \\
& \underline{a}^m \leq z^m + \sum_{t'=1}^t (x_{t'-TL(m)}^m - d_{t'-TM(m)}) \\
& \bar{a}^m \geq z^m + \sum_{t'=1}^t (x_{t'-TL(m)}^m - d_{t'-TM(m)}) \\
& \left. \begin{aligned}
& x_t^j \leq b^j \\
& x_t^j \geq 0 \\
& w_t^j \geq 0
\end{aligned} \right\} \forall j \in \{1, \dots, m\}
\end{aligned} \right\} \forall t \in \{1, \dots, n\} \tag{4}
\end{aligned}$$

As we have already stated, Love (1979) was one of the first researchers to address the issue of controlling inventory fluctuations. Love presented a model for the dynamics of a multi-echelon serial supply chain much like the one described above in equation (1). In the same book he gives an example of controlling this system using a simple control law, that in its general form can be represented as

$$x_t^j = x_{t-1}^{j+1} + \frac{1}{2} (\Upsilon^j - y_{t-1}^j) \quad \begin{aligned} & \forall j \in 1, \dots, m \\ & \forall t \in 1, \dots, n \end{aligned} \tag{5}$$

Here  $x_t^m = d_t$ , and  $\Upsilon^j$  is the “target” inventory for echelon  $j$  (equal to 12, for all the echelons, in the example) and acts as insurance against unforeseen production or supply disruptions. He claims that “This control may not be the most effective one, but it illustrates features of multi-echelon control”. Using his example, he showed the effect of such a strategy on the performance of the chain.

The example uses a fluctuating demand that is shown in table 1<sup>2</sup>. In this table,  $t$  denotes the time period and  $d_t$  the demand of period  $t$ . The planning horizon consists of  $n = 20$  time periods and there are  $m - 3$  echelons. Furthermore, there is a lead time of 2 periods between the time an order departs from its origin and the time it is received at its destination ( $L^j=2$  for all  $j$ ). The initial Inventory is assumed to be 12 for all echelons ( $z^j = 12$  for all  $j$ ).

Using this example, Love shows that employing such a control method causes “inventory levels (to) vary more widely than the demand themselves, and the larger the number of echelons between an inventory and the demand the widely the level of inventory fluctuates”. This can be seen in

<sup>2</sup> Love’s original example contained a 25-period planning horizon, in order to reduce to length of the horizon for computational reasons we truncated the first two and last three periods, keeping the behavior of the demand unchanged.

**Table 1** Love's example demand

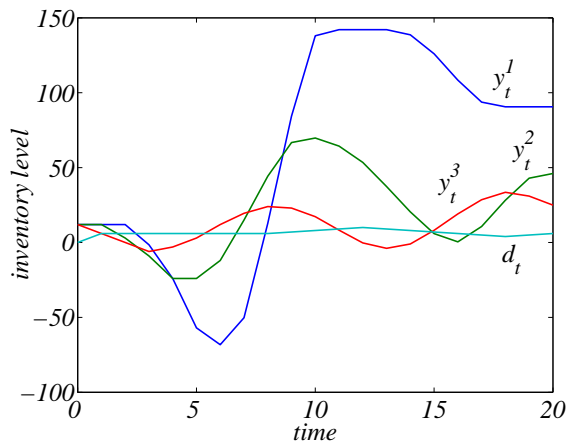
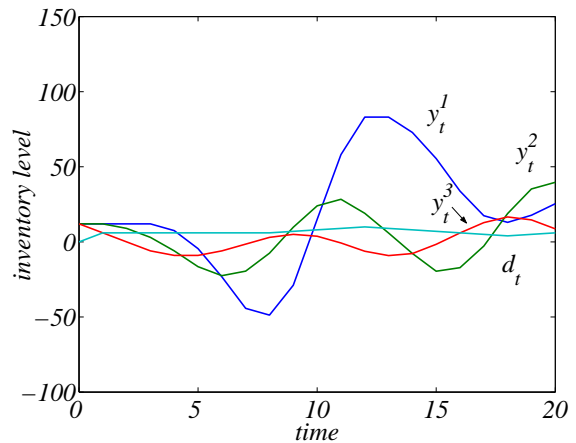
$t$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$d_t$	6	6	6	6	6	6	6	6	7	8	9	10	9	8	7	6	5	4	5	6

figure 2 which shows the inventory levels resulting from implementing this control. He notes that similar fluctuations occur in the ordering rates. Though cost is not explicitly addressed in this example, the inventory fluctuations cause high inventory and shortage costs.

This formulation uses a cost minimization objective. If we choose arbitrary costs such as  $c_t^j = 2$ ,  $h_t^j = 1$  and  $p_t^j = 3$  for all  $t$  and  $j$ , this example yields a "benchmark" cost of 4795 which, as we will see later on, is quite high. We will refer to this method *Love's method* (LV).

The LV method assumes a predetermined target inventory  $\Upsilon^j$ . We can improve this method by treating  $\Upsilon^j$  as a non-negative variable and finding for it an optimal value for which this control strategy will yield a minimal objective function value. We will refer to this method as the *improved Love method* (ILV).

The inventory levels which result from the ILV method applied on Love's example can be seen in figure 3. We note that the amplification is still apparent but not as much as in the original results. Furthermore the objective function value (for the same cost parameter assumed) decreases to 3140.

**Figure 2** Inventory behavior in Love's example**Figure 3** Inventory behavior for the ILV method

The control methods discussed above assume a deterministic demand. As such, they are not equipped to deal with uncertain (stochastic) demand and a new control method is needed.

## 2.2. Dealing with Uncertainty: The Robust Optimization (RO) Approach.

The supply chain problem is an uncertain LP problem, due to the fact that the decision of each echelon, in each period, depends on demands/supplies in the future, and these are typically unknown at the time decisions have to be made. Classical methods dealing with uncertain mathematical programming problems are Stochastic Dynamic Programming (SDP) and Stochastic Programming with Recourse (SPR). In both methods the expected value of the objective function is optimized, and hence knowledge of the probability distribution functions of all the uncertain parameters is required. This imposes a heavy burden on the user. Also, both SDP and SPR become severely computationally intractable even for medium scale multi-period uncertain LPs. A third classical method for dealing uncertain LPs is Chance Constraints Programming, which again needs full

stochastic information and generally leads to nonconvex programs. Robust Optimization (RO) is a methodology that attempts to avoid the above difficulties. We are about to briefly outline the essential ingredients of this methodology, in particular those relevant to our supply chain problem.

**2.2.1. The Robust counterpart of Uncertain LP** Consider a general LP problem:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} c[\lambda]x + d[\lambda] \\ & \text{s.t.} \\ & A_i[\lambda]x - b_i[\lambda] \leq 0 \quad i = 1, \dots, m \end{aligned} \tag{6}$$

where the data  $C[\cdot]$ ,  $A_i[\cdot]$ , and  $b_i[\cdot]$  depend on uncertain parameter vector  $\lambda$  which can range in an *uncertain set*  $\mathcal{U}$  - a convex compact set. Here we assume that all the above functions of  $\lambda$  are affine (e.g.  $c[\lambda] = c_0 + C\lambda$  for some fixed vector  $c_0$  and matrix  $C$ ). The notation  $[\lambda]$  is used to indicate affine dependence on  $\lambda$ .

An "uncertain LP" is then a family of LPs, one for each instance of  $\lambda \in \mathcal{U}$ . The *Robust Counterpart* (RC) of this uncertain LP is defined as follows:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n, \tau \in \mathbb{R}} \tau \\ & \text{s.t.} \\ & \left. \begin{aligned} c[\lambda]x + d[\lambda] &\leq \tau \\ A_i[\lambda]x - b_i[\lambda] &\leq 0 \quad i = 1, \dots, m \end{aligned} \right\} \forall \lambda \in \mathcal{U} \end{aligned} \tag{7}$$

A solution  $(\bar{x}, \bar{\tau})$  of (7) satisfies the constraints of (6) for every  $\lambda \in \mathcal{U}$  (i.e. it is *robust feasible* and among such robust feasible solutions  $\bar{\tau}$  is the best possible objective function of (6) (i.e.  $\bar{\tau}$  is the *robust optimal value*).

The RC problem (7) is a single deterministic problem, however with a continuum of constraints. Nevertheless, the theory developed by Ben-Tal and Nemirovski (2002) shows that (7) is computationally tractable (polynomially solvable) for a wide choice of the uncertainty set  $\mathcal{U}$ . In particular, if  $\mathcal{U}$  is a polyhedral set, then the RC (7) is itself a LP. If  $\mathcal{U}$  is an intersection of ellipsoids and polyhedrons then the RC (7) is a *conic quadratic program*, which is indeed polynomially solvable (by Interior Point methods) and for which commercial software is available.

**2.2.2. The Adjustable Robust Counterpart of Multi Period Uncertain LP** The RO paradigm assumes implicitly that all the decision variables are determined **prior** to the realization of the uncertainty ("here and now" decisions). In a dynamical (multi-periods) problem, such as our supply chain problem, this is not the case; decisions of each echelon related to period  $t > 1$  can be postponed to that period, at which time the demands in periods  $1, 2, \dots, t-1$  are known ("wait and see" decisions). Thus  $x_t$  - the vector of decisions at time  $t$ , should be a function of (possible part of) the historical data  $\lambda_1, \lambda_2, \dots, \lambda_{t-1}$ :

$$x_t = x_t(P_t \lambda) \tag{8}$$

where the matrix  $P_t$  determines on what part of the full data vector  $\lambda_1, \lambda_2, \dots, \lambda_T$   $x_t$  will depend. E.g. if  $x_t$  is chosen to depend on the entire past  $\lambda_1, \lambda_2, \dots, \lambda_{t-1}$  then:

$$P_t = [I_{t-1} | 0_{(t-1) \times T}]$$

where  $I_{t-1}$  is the  $(t-1) \times (t-1)$  identity matrix and  $0_{(t-1) \times T}$  is the  $(t-1) \times T$  matrix with all entries equal to zero.

Decision variables given as in (8) for which  $P_t \neq 0$  are called *adjustable*, those with  $P_t = 0$  are nonadjustable. The adjustable variables are in fact *policies*, they admit a numerical value only when the part of  $\lambda_1, \lambda_2, \dots, \lambda_T$  on which they depend becomes available.

For LPs with adjustable variables, the notion of RC is generalized to the following *Adjustable Robust Counterpart* (ARC):

$$\begin{aligned} & \min_{x_t(\cdot), \tau} \tau \\ & \text{s.t.} \\ & \left. \begin{aligned} & \sum_t c_t[\lambda] x_t(P_t \lambda) + d[\lambda] \leq \tau \\ & \sum_t A_{it}[\lambda] x_t(P_t \lambda) - b_i[\lambda] \leq 0 \quad i = 1, \dots, m \end{aligned} \right\} \forall \lambda \in \mathcal{U} \end{aligned} \quad (9)$$

The control variables are now the real  $\tau$  and the functions  $x_t(\cdot)$ .

Unfortunately the ARC problem as given in (9) is NP-hard even for trivial choices of the uncertainty set  $\mathcal{U}$  (see Ben-Tal et al. (2004)). In the latter case, the optimal policies can be computed by dynamic programming.

To avoid this computational trap, Ben-Tal et al. (2004) proposed to restrict the controls to be affine functions, i.e.:

$$x_t(\lambda) = x_t^0 + X_t \lambda \quad t = 1, 2, \dots, T \quad (10)$$

the decision variables then become the vector  $x_t^0$  and the matrix  $X_t$ . When substituting (10) in the ARC formulation (9), it becomes a regular RC (though of larger dimensions) which is called *Affine ARC* (AARC).

In our supply chain problem the uncertainty lies only in  $b_i[\lambda]$ , so the parameters multiplying variables  $x_t(\cdot)$  are fixed. such problems are said to be with *fixed recourse*. For uncertain problems with fixed recourse the entire theory of regular RC is valid, and such problems are then computationally tractable for a wide spectrum of uncertainty sets  $\mathcal{U}$ .

**2.2.3. The Generalized Robust Counterpart of Uncertain LP** Guaranteeing feasibility of a constraint for *all physically possible* instances, even rare ones, may require a large uncertainty set, and as a result overly conservative decisions. The *Globalized Robust Counterpart* (GRC), developed by Ben-Tal et al. (2005), addresses this issue by requiring feasibility on a subset  $\mathcal{U}$  of all physically possible instances, that includes the "normal range" of the uncertain parameters. For events outside  $\mathcal{U}$ , infeasibility is tolerated, but it is controlled.

Consider a single uncertain linear constraint:

$$a[\lambda]^T x - b[\lambda] \leq 0 \quad (11)$$

Let  $\mathcal{U}$  be the normal range of the uncertain parameter vector  $\lambda$ .

The GRC of (11) is as follows:

$$a[\lambda]^T x - b[\lambda] \leq \alpha \text{dist}(\lambda, \mathcal{U}) \quad (12)$$

where  $\text{dist}(\lambda, \mathcal{U}) \equiv \inf_{u \in \mathcal{U}} \|\lambda - u\|$  and  $\alpha \geq 0^3$ .

Note that whenever  $\lambda \in \mathcal{U}$  then  $\text{dist}(\lambda, \mathcal{U}) = 0$  and hence (11) is indeed **feasible**  $\forall \lambda \in \mathcal{U}$ .

When  $\lambda \notin \mathcal{U}$ ,  $\text{dist}(\lambda, \mathcal{U}) > 0$  and so (11) may be infeasible for such  $\lambda$  (and more so the further  $\lambda$  from  $\mathcal{U}$ , i.e. when  $\lambda$  is a "rare event"), but the infeasibility is controlled by the parameter  $\alpha > 0$ . When  $\alpha = 0$  (12) implies that (11) is feasible even for  $\lambda \notin \mathcal{U}$ ; this choice of  $\alpha$  reflects very conservative decision makers. On the extreme,  $\alpha = \infty$ , (12) implies that (11) holds for  $\lambda \in \mathcal{U}$ , but **nothing is required** from the decisions vector  $x$  when  $\lambda \notin \mathcal{U}$ , this choice of  $\alpha$  then represents a somewhat "irresponsible" decision maker.  $\alpha$  can be either a predetermined parameter or a variable. In the latter case one could add constraints on  $\alpha$  to limit its range or add some function of  $\alpha$  to the objective function for the same purpose.

<sup>3</sup> Here, for simplicity, we take the whole space  $\mathbb{R}^T$  as the set of physically possible values of  $\lambda$



The GRC of uncertain LP (6) is then:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n, \tau \in \mathbb{R}} \tau \\ & s.t. \left. \begin{aligned} c[\lambda]x + d[\lambda] - \tau &\leq \alpha_0 \cdot \text{dist}(\lambda, \mathcal{U}) \\ A_i[\lambda]x - b_i[\lambda] &\leq \alpha_i \cdot \text{dist}(\lambda, \mathcal{U}) \quad i = 1, \dots, m \end{aligned} \right\} \forall \lambda \in \mathbb{R}^T \end{aligned} \quad (13)$$

It is proved in Ben-Tal et al. (2005), that when the norms defining the distance function  $\text{dist}(\lambda, \mathcal{U})$  are polyhedral (e.g.  $\ell_1$  or  $\ell_\infty$  norms), and the normal range uncertainty set  $\mathcal{U}$  is polyhedral, then GRC in (13) is reducible to a simple LP.

### 2.3. Supply Chain Control as a General Control Problem.

In the heart of the supply chain problem (3) resides a discrete-time linear control system of the generic representation:

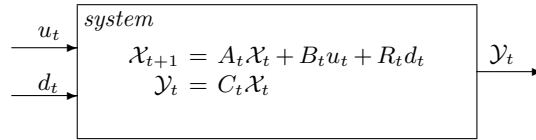
$$\left. \begin{aligned} \mathcal{X}_{t+1} &= A_t \mathcal{X}_t + B_t u_t + R_t d_t \\ \mathcal{Y}_t &= C_t \mathcal{X}_t \\ \mathcal{X}_0 &= z \end{aligned} \right\} \forall t \in \{1 \dots n\} \quad (14)$$

where  $z$  is the initial state of the system and for each time  $t$ :

- $\mathcal{X}_t$  is the state of the system.
- $u_t$  is the control.
- $d_t$  is the exogeneous input.
- $\mathcal{Y}_t$  is the observable output.

and  $A_t$ ,  $B_t$ ,  $C_t$  and  $R_t$  are given matrices of appropriate sizes. Figure 4 describes the dynamics of a general control system of this form.

**Figure 4** Open loop Control



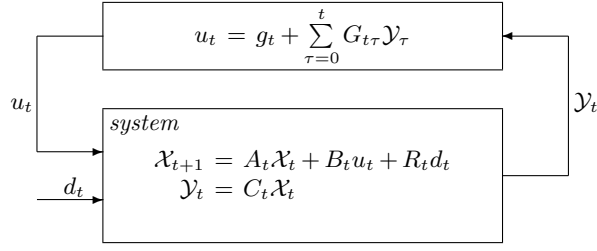
A widely used control law for this type of systems is a *non-anticipating linear control law*, which depends affinely on the observed outputs:

$$u_t = g_t + \sum_{\tau=0}^t G_{t\tau} \mathcal{Y}_\tau \quad \forall t \in \{1 \dots n\} \quad (*)$$

where  $g_t$  and  $G_{t\tau}$  are vectors and matrices of appropriate sizes. The closed loop system generated through this control law is shown in figure 5.

In our control problem  $\mathcal{X}_t$  is the vector of inventory levels of all the echelons at time  $t$ , and  $u_t$  is the vector of orders submitted by the echelons at time  $t$ . Our control problem has then the structure:

$$\left. \begin{aligned} \mathcal{X}_{t+1} &= A_t \mathcal{X}_t + \sum_{\tau=t-T^L}^t B_{t\tau} u_\tau + R_t d_t \\ \mathcal{Y}_t &= \mathcal{X}_t \\ \mathcal{X}_0 &= z \end{aligned} \right\} \forall t \in \{1 \dots n\} \quad (15)$$

**Figure 5** Closed loop system

which is slightly different from the one described in (14). To convert problem (15) to the form given in (14) we add  $\mathcal{Z}_t^\tau$  to the state variables of time  $t$  and rewrite system (15) as:

$$\left. \begin{aligned} \mathcal{X}_{t+1} &= A_t \mathcal{X}_t + \sum_{\tau=t-T^L}^{t-1} B_{t\tau} \mathcal{Z}_t^\tau + B_{tt} u_t + R_t d_t \\ \mathcal{Z}_{t+1}^\tau &= \mathcal{Z}_t^\tau \quad \forall \tau \in \{t-T^L+1, \dots, t-1\} \\ \mathcal{Z}_{t+1}^t &= u_t \\ \mathcal{Y}_t &= \mathcal{X}_t \\ \mathcal{X}_0 &= z \\ \mathcal{Z}_0^\tau &= 0 \quad \forall \tau \in \{-(T^L+1), \dots, -1\} \end{aligned} \right\} \forall t \in \{1 \dots n\} \quad (16)$$

This latter system is then of the generic form (14) with  $C_t$  being the identity matrix so that  $\mathcal{Y}_t = \mathcal{X}_t$ .

For the generic system (14), Ben-Tal et al. (2005) showed that using the control law (\*) can result in a system control equations that are highly nonlinear in the control coefficients  $\{G_{t\tau}\}$ . This causes the feasible set of the robust counterpart to be non convex, making the problem intractable, and preclude the use of the AARC or the GRC methods. As an example let us view the simple system shown in equation (17).

$$\left. \begin{aligned} \mathcal{X}_{t+1} &= \mathcal{X}_t + u_t + d_t \\ \mathcal{Y}_t &= \mathcal{X}_t \\ \mathcal{X}_0 &= 0 \end{aligned} \right\} t = 1, \dots, n \quad (17)$$

Suppose we use the control (\*) and wish to satisfy a system of linear inequalities:

$$A\mathcal{W} \geq b \quad (18)$$

where  $\mathcal{W} = (\mathcal{X}, u)$  is the system trajectory. Then:

$$\begin{aligned} \mathcal{X}_0 &= 0 \\ u_0 &= g_0 \\ \mathcal{X}_1 &= \mathcal{X}_0 + u_0 + d_0 = g_0 + d_0 \\ u_1 &= g_1 + G_{11} \mathcal{X}_1 = g_1 + G_{11} g_0 + G_{11} d_0 \\ \mathcal{X}_2 &= \mathcal{X}_1 + u_1 + d_1 = g_0 + d_0 + g_1 + G_{11} g_0 + G_{11} d_0 + d_1 \\ u_2 &= g_2 + G_{21} \mathcal{X}_1 + G_{22} \mathcal{X}_2 = g_2 + G_{21} g_0 + G_{21} d_0 + G_{22} (g_0 + d_0 + g_1 + G_{11} g_0 + G_{11} d_0 + d_1) \\ &= d_1 G_{22} + d_0 (G_{21} + G_{22} + G_{22} G_{11}) + g_2 + G_{21} g_0 + G_{22} g_0 + G_{22} g_1 + G_{22} G_{11} g_0 \end{aligned}$$

We can clearly see that the trajectory is linear in  $(d_0, d_1)$  but non-linear (non convex) in the  $G_{t\tau}$ s

To avoid nonconvexity and achieve bi-affinity (i.e. affinity in **both** w.r.t  $d_t$  and the control coefficients) Ben-Tal et al. (2005) employ a reparameterization scheme and use the following control law:

$$u_t = h_t + \sum_{\tau=0}^t H_{t\tau} v_\tau \quad \forall t \in \{1 \dots n\} \quad (**)$$

Here  $h_t$  and  $H_{t\tau}$  are vectors and matrices of appropriate sizes, and the new parameters  $v_t$  are derived from the solution of the following "purified" system:

$$\left. \begin{aligned} \widehat{\mathcal{X}}_{t+1} &= A_t \widehat{\mathcal{X}}_t + B_t u_t \\ \widehat{\mathcal{Y}}_t &= C_t \widehat{\mathcal{X}}_t \\ \widehat{\mathcal{X}}_0 &= 0 \end{aligned} \right\} \forall t = 1, \dots, n \quad (19)$$

specifically:

$$v_t = \mathcal{Y}_t - \widehat{\mathcal{Y}}_t$$

In (19)  $\widehat{\mathcal{X}}$  and  $\widehat{\mathcal{Y}}$  are the state and the output of the purified system, respectively, obtained by using the same control as in the original system (14). Note that system (19) is obtained from (14) by assigning a value of zero to the initial state and the time dependent inputs  $d_t$  (thus eliminating their influence). The new parameters  $v_t$  are called *purified outputs*.

This re-parameterization, which is shown to be bi-linear in the uncertain parameters and control variables.

In our case  $\widehat{\mathcal{Y}}_t = \widehat{\mathcal{X}}_t$ , hence the required control system is:

$$\left. \begin{aligned} \mathcal{X}_{t+1} &= A_t \mathcal{X}_t + B_t u_t + D_t d_t \\ \mathcal{X}_0 &= z \end{aligned} \right\} \quad (20)$$

and its corresponding purified system is:

$$\left. \begin{aligned} \widehat{\mathcal{X}}_{t+1} &= A_t \widehat{\mathcal{X}}_t + B_t u_t \\ \widehat{\mathcal{X}}_0 &= 0 \end{aligned} \right\} \quad (21)$$

the purified outputs are given by  $v_t = \mathcal{X}_t - \widehat{\mathcal{X}}_t$ .

Let us consider again problem (18) using, this time, the purified-based controller (\*\*). We obtain the following:

$$\begin{aligned} \mathcal{X}_0 &= 0 \\ \widehat{\mathcal{X}}_0 &= 0 \\ v_0 &= \mathcal{X}_0 - \widehat{\mathcal{X}}_0 = 0 \\ u_0 &= h_0 \\ \mathcal{X}_1 &= \mathcal{X}_0 + u_0 + d_0 = h_0 + d_0 \\ \widehat{\mathcal{X}}_1 &= \widehat{\mathcal{X}}_0 + u_0 = h_0 \\ v_1 &= \mathcal{X}_1 - \widehat{\mathcal{X}}_1 = d_0 \\ u_1 &= h_1 + H_{11} v_1 = h_1 + H_{11} d_0 \\ \mathcal{X}_2 &= \mathcal{X}_1 + u_1 + d_1 = h_0 + d_0 + h_1 + H_{11} d_0 + d_1 \\ \widehat{\mathcal{X}}_2 &= \widehat{\mathcal{X}}_1 + u_1 = h_0 + h_1 + H_{11} d_0 \\ v_2 &= \mathcal{X}_2 - \widehat{\mathcal{X}}_2 = d_0 + d_1 \\ u_2 &= h_2 + H_{21} v_1 + H_{22} v_2 = h_2 + H_{21} d_0 + H_{22} (d_0 + d_1) \end{aligned}$$

This trajectory is obviously linear in both  $(d_0, d_1)$  and  $H_{t\tau}$ s, and so any linear constraints (such as (18)) will be indeed bi-affine!

This result was proven in Ben-Tal et al. (2005) for the general dynamic system (14). Here we recapture the proof for the specific case of (14) with  $C_t = I$  (i.e.  $\mathcal{Y}_t = \mathcal{X}_t$ ).

**PROPOSITION 2.1.** (i) For every control law  $u_t$  of the form  $u_t = g_t + \sum_{\tau=0}^t G_{t\tau} \mathcal{X}_\tau$  there is an equivalent control law of the form  $u_t = h_t + \sum_{\tau=0}^t H_{t\tau} v_\tau$ , i.e. whatever be the initial state  $z$  and sequence of inputs  $d_t$ , they will both result in exactly the same state-control trajectories of the closed loop system.

(ii) Vice Versa, for every control law  $u_t$  of the form  $u_t = h_t + \sum_{\tau=0}^t H_{t\tau} v_\tau$  there is an equivalent control law of the form  $u_t = g_t + \sum_{\tau=0}^t G_{t\tau} \mathcal{X}_\tau$ , i.e. whatever be the initial state and sequence of inputs, they will both result in exactly the same state-control trajectories of the closed loop system.

(iii) (bi-affinity) The state-control trajectory  $\mathcal{W}^n = (\mathcal{X}^{n+1} = \{\mathcal{X}_0, \dots, \mathcal{X}_{n+1}\}, u^n = \{u_0, \dots, u_n\})$  of the closed loop system is affine in  $z$  and  $d^n = \{d_1, \dots, d_n\}$  when the parameters  $\eta = \{h_t, H_{t\tau}\}_{0 \leq \tau \leq t \leq n}$  of the underlying control law are fixed, and is affine in  $\eta$  when  $z$  and  $d^n$  are fixed.

**Proof of proposition 2.1.**

(i) In order to prove that for every control law of the form  $u_t = g_t + \sum_{\tau=0}^t G_{t\tau} \mathcal{X}_\tau$  there exists an equivalent control law of the form  $u_t = h_t + \sum_{\tau=0}^t H_{t\tau} v_\tau$  it suffices to show that  $\mathcal{X}_t = q_t + \sum_{\tau=0}^t Q_{t\tau} v_\tau$  for some  $Q_{t\tau}$ .

Proof by induction. It is easy to see  $v_0 = \mathcal{X}_0 - \widehat{\mathcal{X}}_0 = \mathcal{X}_0 = z$ . Let us assume that for each  $0 \leq t \leq s$  that  $\mathcal{X}_t = q_t + \sum_{\tau=0}^t Q_{t\tau} v_\tau$ , and try to prove it for  $s+1$ . Now,  $\mathcal{X}_{s+1} = A_s \mathcal{X}_s + B_s u_s + D_s d_s$  but  $u_s$  is known to be affine in  $\mathcal{X}^s = \{\mathcal{X}_0, \dots, \mathcal{X}_s\}$  and  $\mathcal{X}_t$  is affine in  $v^t = \{v_0, \dots, v_t\}$  for all  $0 \leq t \leq s$ , thus  $\mathcal{X}_{s+1}$  is affine in  $v^s$  and more generally in  $v^{s+1}$  and is of the form  $\mathcal{X}_{s+1} = q_{s+1} + \sum_{\tau=0}^{s+1} Q_{(s+1)\tau} v_\tau$ . Induction is complete and (i) is proven.

(ii) In order to prove that for every control law of the form  $u_t = h_t + \sum_{\tau=0}^t H_{t\tau} v_\tau$  there exists an equivalent control law of the form  $u_t = g_t + \sum_{\tau=0}^t G_{t\tau} \mathcal{X}_\tau$ , it suffices to show that  $v_t = p_t + \sum_{\tau=0}^t P_{t\tau} \mathcal{X}_\tau$  for some  $P_{t\tau}$ .

Proof by induction. as we have shown  $v_0 = \mathcal{X}_0$ . Let us assume that for each  $0 \leq t \leq s$  that  $v_t = p_t + \sum_{\tau=0}^t P_{t\tau} \mathcal{X}_\tau$ , and try to prove it for  $s+1$ . Now,  $v_{s+1} = \mathcal{X}_{s+1} - \widehat{\mathcal{X}}_{s+1} = A_s \mathcal{X}_s + B_s u_s + R_s d_s - (A_s \widehat{\mathcal{X}}_s + B_s u_s)$  hence  $v_{s+1} = A_s (\mathcal{X}_s - \widehat{\mathcal{X}}_s) + R_s d_s = A_s v_s + R_s d_s$  since we know  $v_s$  is affine in  $\mathcal{X}^s = \{\mathcal{X}_0, \dots, \mathcal{X}_s\}$  so is  $v_{s+1}$  and more generally it is also affine in  $\mathcal{X}^{s+1}$  and is of the form  $v_{s+1} = p_{s+1} + \sum_{\tau=0}^{s+1} P_{s+1\tau} \mathcal{X}_\tau$ . Induction is complete and (ii) is proven.

(iii) We have shown in (ii) that  $v_{t+1} = A_t v_t + R_t d_t$  and  $v_0 = z$  so

$$v_{t+1} = A_0^t z + \sum_{i=1}^{t+1} A_i^t R_{i-1} d_{i-1}$$

where

$$A_i^t = \begin{cases} \prod_{j=i}^t A_j, & i \leq t \\ I, & i = t+1 \end{cases}$$

Therefore the control law implies that,

$$\begin{aligned} u_t &= h_t + \sum_{\tau=0}^t H_{t\tau} v_\tau = h_t + \sum_{\tau=0}^t H_{t\tau} A_0^{\tau-1} z + \sum_{i=1}^{\tau} A_i^{\tau-1} R_{i-1} d_{i-1} \\ &= h_t + \underbrace{\left[ \sum_{\tau=0}^t H_{t\tau} A_0^{\tau-1} \right]}_{N_t(\eta)} z + \sum_{i=1}^t \underbrace{\left[ R_{i-1} \sum_{\tau=i}^t H_{t\tau} A_i^{\tau-1} \right]}_{N_{ti}(\eta)} d_{i-1} \end{aligned}$$

showing that  $u_t$  is bi-affine in  $\eta$  (the vector of coefficients  $H_{t\tau}$ ) and  $(z, d^n$  (hereinafter bi-affine)). To prove that  $\mathcal{X}_t$  is also bi-affine we will use induction.  $\mathcal{X}_0 = z$  is bi affine. Let us assume that  $\mathcal{X}_t$  is bi-affine for all  $0 \leq t \leq s$  and prove it for  $s+1$ . Since  $\mathcal{X}_{s+1} = A_s \mathcal{X}_s + B_s u_s + R_s d_s$  and  $\mathcal{X}_s$  is bi-affine according to the induction assumption, and  $u_s$  is bi-affine by the previous argument we can conclude that  $\mathcal{X}_{s+1}$  is bi-affine as well. Induction is complete. Therefore, we showed that both

the state  $\mathcal{X}_t$  ( $0 \leq t \leq n+1$ ) and the control  $u_t$  ( $0 \leq t \leq n$ ) of the closed loop system are bi-affine. ■

Thanks to the results expressed in proposition 2.1 we now have a efficient way to control linear systems by a purified output-based linear control law. The next step will be to combine this control methodology with the RO methodology to obtain robust control of linear dynamic systems. In our formulation of the supply chain problem we also have constraints on both the state and control variables. This, however, does not cause any difficulty and bi-affinity results are maintained.

#### 2.4. The Supply Chain Problem as a Control Problem Affected by Uncertainty

Returning to our supply chain problem (3) let us recall the inventory (state) orders (control) relationship:

$$\begin{aligned} y_t^j &= y_{t-1}^j + x_{t-(I(j)+M(j-1)+L(j))}^j - x_{t-(I(j+1)+M(j))}^{j+1} & \forall j \in \{1, \dots, m-1\} \\ y_0^j &= z^j & \forall j \in \{1, \dots, m\} \\ y_t^m &= y_{t-1}^m + x_{t-(I(m)+M(m-1)+L(m))}^m - d_{t-(I(m+1)+M(m))} & \end{aligned}$$

Here the purified output are given by:

$$v_t^j = y_t^j - \hat{y}_t^j = \begin{cases} z^m - \sum_{\tau=1}^{t-T^M(m)} d_\tau & \text{if } j = m \\ z^j & \text{otherwise} \end{cases} \quad (22)$$

After eliminating the equalities in (3) we arrived at the LP problem (4) which we showed to be of a form amenable to treatment by the RO methodology. We use a purified output based linear control law and make also associated auxiliary variables affinely dependent on the uncertain data, specifically:

$$\begin{aligned} x_t^j &\equiv x_t^j(d, z) = \bar{\eta}_0^{x,t,j} + \sum_{\substack{j' \in \{1, \dots, m\} \\ \tau = \{1, \dots, n\}}} \eta_{\tau, j'}^{x,t,j} z^{j'} - \sum_{\substack{\tau = \{1, \dots, n\} \\ \tau' = \{1, \dots, \tau - M(m)\}}} \eta_{\tau, m}^{x,t,j} d_{\tau'} \\ w_t^j &= \bar{\eta}_0^{w,t,j} + \sum_{j' \in \{1, \dots, m\}} \tilde{\eta}_{j'}^{w,t,j} z^{j'} + \sum_{\tau = \{1, \dots, T\}} \eta_\tau^{w,t,j} d_\tau \end{aligned} \quad (23)$$

Of course we impose the constraints  $\eta_{\tau, j'}^{x,t,j} = 0 \quad \forall \tau \geq t$  and set  $\eta_\tau^{w,t,j} = 0 \quad \forall \tau \geq t$  so that the affine rule is non-anticipative.

We see that the method we arrive at is in fact the AARc method introduced in section 2.4. Moreover the AARC is indeed bi-affine and so it is amenable to treatment by the GRC methodology.

Let us take the second constraint from equation (4) to illustrate the implementation of the linear controller and then the GRC. As we recall the original constraint is of the form :

$$w_t^j \geq h_t^j(z^j + \sum_{t'=1}^t (x_{t'-T^L(j)}^j - x_{t'-T^M(j)}^{j+1})) \quad (24)$$

Implementing the linear equalities shown in (23) will arrive at:

$$\begin{aligned} h_t^j &\sum_{t'=1, \dots, t} (\bar{\eta}_0^{x,t'-T^L(j),j} - \bar{\eta}_0^{x,t'-T^M(j),j+1}) - \bar{\eta}_0^{w,t,j} + h_t^j z^j + \\ &\sum_{j'=1, \dots, m} z^{j'} [h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=1, \dots, n}} (\eta_{\tau, j'}^{x,t'-T^L(j),j} - \eta_{\tau, j'}^{x,t'-T^M(j),j+1}) - \tilde{\eta}_{j'}^{w,t,j}] + \\ &\sum_{\tau'=1, \dots, n} d_{\tau'} [-h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=\tau'+M(m), \dots, n}} (\eta_{\tau', m}^{x,t'-T^L(j),j} - \eta_{\tau', m}^{x,t'-T^M(j),j+1}) - \eta_{\tau'}^{w,t,j}] \leq 0 \end{aligned} \quad (25)$$

We now implement the GRC assuming that the normal ranges of both  $d_t$  and  $z^j$  are the intervals  $[\underline{d}_t, \bar{d}_t]$  and  $[\underline{z}^j, \bar{z}^j]$  respectively and the norms defining the distance function in (13) is the  $\ell_1$ -norm. Thus it can be shown (based on Ben-Tal et al. (2005)) that the GRC of (25) is the linear system below.

$$\begin{aligned}
& h_t^j \sum_{t'=1, \dots, t} (\bar{\eta}_0^{x, t' - T^L(j), j} - \bar{\eta}_0^{x, t' - T^M(j), j+1}) - \bar{\eta}_0^{w, t, j} + \sum_{j'=1, \dots, m} v_{j'}^{2, t, j} + \sum_{\tau'=1, \dots, n} \vartheta_{\tau'}^{2, t, j} \leq 0 \\
& \underline{z}^{j'} [h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=1, \dots, n}} (\eta_{\tau, j'}^{x, t' - T^L(j), j} - \eta_{\tau, j'}^{x, t' - T^M(j), j+1}) - \tilde{\eta}_{j'}^{w, t, j} + h_t^j I_{\{j'=j\}}] \leq v_{j'}^{2, t, j} \quad \forall j' = 1, \dots, m \\
& \bar{z}^{j'} [h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=1, \dots, n}} (\eta_{\tau, j'}^{x, t' - T^L(j), j} - \eta_{\tau, j'}^{x, t' - T^M(j), j+1}) - \tilde{\eta}_{j'}^{w, t, j} + h_t^j I_{\{j'=j\}}] \leq v_{j'}^{2, t, j} \quad \forall j' = 1, \dots, m \\
& \underline{d}_{\tau'} [-h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=\tau'+M(m), \dots, n}} (\eta_{\tau, m}^{x, t' - T^L(j), j} - \eta_{\tau, m}^{x, t' - T^M(j), j+1}) - \eta_{\tau'}^{w, t, j}] \leq \vartheta_{\tau'}^{2, t, j} \quad \forall \tau' = 1, \dots, n \\
& \bar{d}_{\tau'} [-h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=\tau'+M(m), \dots, n}} (\eta_{\tau, m}^{x, t' - T^L(j), j} - \eta_{\tau, m}^{x, t' - T^M(j), j+1}) - \eta_{\tau'}^{w, t, j}] \leq \vartheta_{\tau'}^{2, t, j} \quad \forall \tau' = 1, \dots, n \\
& [h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=1, \dots, n}} (\eta_{\tau, j'}^{x, t' - T^L(j), j} - \eta_{\tau, j'}^{x, t' - T^M(j), j+1}) - \tilde{\eta}_{j'}^{w, t, j} + h_t^j I_{\{j'=j\}}] \leq \mu_{j'}^{2, t, j} \quad \forall j' = 1, \dots, m \\
& -[h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=1, \dots, n}} (\eta_{\tau, j'}^{x, t' - T^L(j), j} - \eta_{\tau, j'}^{x, t' - T^M(j), j+1}) - \tilde{\eta}_{j'}^{w, t, j} + h_t^j I_{\{j'=j\}}] \leq \mu_{j'}^{2, t, j} \quad \forall j' = 1, \dots, m \\
& [-h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=\tau'+M(m), \dots, n}} (\eta_{\tau, m}^{x, t' - T^L(j), j} - \eta_{\tau, m}^{x, t' - T^M(j), j+1}) - \eta_{\tau'}^{w, t, j}] \leq \tilde{\mu}_{\tau'}^{2, t, j} \quad \forall \tau' = 1, \dots, n \\
& -[-h_t^j \sum_{\substack{t'=1, \dots, t \\ \tau=\tau'+M(m), \dots, n}} (\eta_{\tau, m}^{x, t' - T^L(j), j} - \eta_{\tau, m}^{x, t' - T^M(j), j+1}) - \eta_{\tau'}^{w, t, j}] \leq \tilde{\mu}_{\tau'}^{2, t, j} \quad \forall \tau' = 1, \dots, n
\end{aligned} \tag{26}$$

Here  $\mu^{2, t, j}$  and  $\tilde{\mu}^{2, t, j}$  are the "sensitivity parameters" (previously denoted by  $\alpha$ ). Rather than choosing them in advance, we treat them here as variables, and to limit their variability we add the following constraints:

$$\begin{aligned}
& \sum_{\substack{t=1, \dots, n \\ j=1, \dots, m}} \mu_{j'}^{2, t, j} \leq \alpha_{2, Z_{j'}} \quad \forall j' = 1, \dots, m \\
& \sum_{\substack{t=1, \dots, n \\ j=1, \dots, m}} \tilde{\mu}_{\tau'}^{2, t, j} \leq \alpha_{2, D_{\tau'}} \quad \forall \tau' = 1, \dots, n
\end{aligned}$$

The complete GRC formulation of the supply chain problem (15) is given in <http://ie.technion.ac.il/Labs/Opt/GRC>; it is a LP with  $O(m^2n + mn^2)$  constraints and  $O(m^2n^2)$  variables (more precisely  $1 + 27m + 27n + mn(8 + 28m + 28n)$  constraints and  $1 + 8m + 8n + 2mn + mn(2 + 15m + 15n + mn)$  variables). As an example for a problem with 3 echelons and 20 time periods, and where all the linear decision rules (LDRs) use the entire demand history, the LP has 39,742 constraints and 24,605 variables. Such a problem is solved in about 900 seconds using a state of the art LP solver on a PC with an AMD processor 1.8Ghz and 1GB of memory.

### 3. Computational Results

In this section we will present the simulated results we obtained by using the robust optimization methods we presented in the section 2. In order to estimate the quality of our solution we will use

the *utopian solution* as a reference. For a given simulated demand trajectory and initial inventory the utopian solution is the solution of the corresponding deterministic LP (4). the utopian optimal value constitutes a lower bound on the cost for the best control method that can be achieved. The particular GRC we used in this section is the one corresponding to the upper bound on  $\alpha$  (obtained by trial and error) which yielded the lowest cost.

We will test the methods over different simulated demands using several effectiveness measures. The two leading effectiveness measures are *cost* and the *deviation ratio*; the latter is the ratio  $R_k^A = (\frac{A_k}{OPT_k} - 1)$  where  $A_k$  is the value of the objective function in the solution for method  $A$  for some realization  $k$  and  $OPT_k$  is the utopian solution for the same realization.

We will use different examples to illustrate different issues related to the GRC solution.

### 3.1. Bullwhip Effect Results.

In the previous section we showed that some control can results in the "amplification of oscillation up the supply chain" also known as the Bullwhip effect. We wanted to examine whether the robust methods will exhibit the same behavior given an oscillating demand.

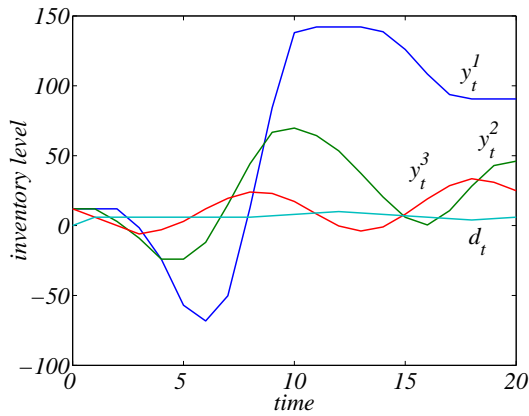
For this purpose we chose the same example given in the section 2.1, a 3 echelon supply chain with planning horizon of 20 periods. All other data is as given in table 2. The actual demand, shown in table 1, is the one used in Love (1979).

**Table 2** Test case - data

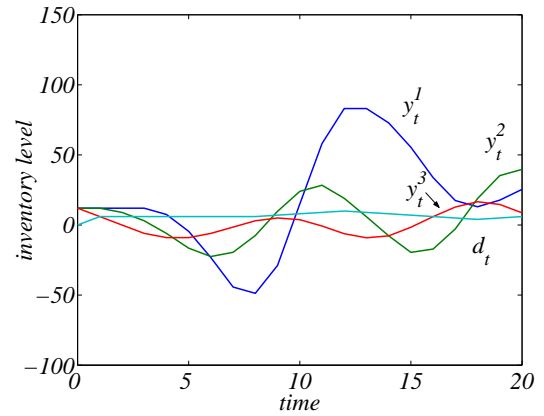
Data type	Notation	Value
General	n	20
	m	3
Delays (for all j)	L(j)	2
	M(j)	0
	I(j)	0
Costs (for all j and t)	$c_j^t$	2
	$h_j^t$	1
	$p_j^t$	3
Normal ranges for uncertainty sets (for all j and t)	$D_t$	[4, 10]
	$Z^j$	[10,14]

The results of the various controls, including those of the LV and ILV methods (shown in section 2.1), those of the various robust methods and those of the utopian solution, are displayed in figures 6-11

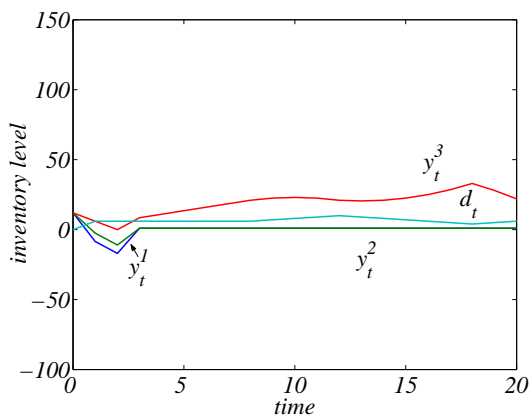
In these figures, the cyan, red, green and blue lines describe the behavior of the demand  $d_t$ , and the inventory levels  $y_t^3$ ,  $y_t^2$  and  $y_t^1$  respectively for the three echelons. We can observe that the LV method exhibits the Bullwhip. This is less extreme but still considerable for the ILV method. For the RC, AARC and GRC strategies, and of course the utopian solution, we do not observe such amplification. It is interesting to note that, contrary to the LV and ILV methods, in the robust solutions the inventory in the retailer level fluctuates *more* than that of the wholesale and manufacturer levels. In the LV method the inventory level in echelon 1, the furthest echelon from the external demand, reaches a minimum value of about -70 units and a maximum value of about 140 units, which is 35 times the range of the external demand (which fluctuates between 4 and 10 units). In the ILV method the same inventory level spans between -50 and 80 units (22 times the range of the external demand). The robust methods are much more stable: the highest inventory level in all the echelon is about 30 (in echelon 3) and the lowest is about -20, and each echelon does not fluctuate more than 5 times the external demand range.



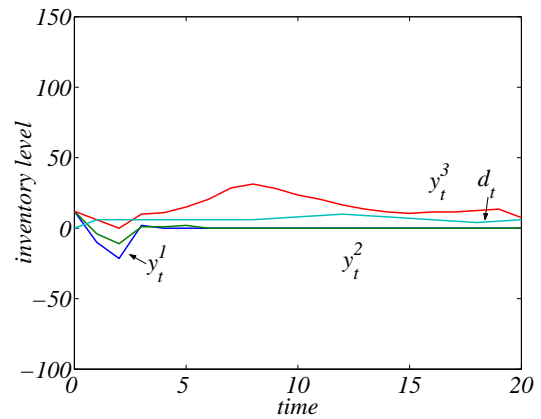
**Figure 6** Inventory behavior - LV



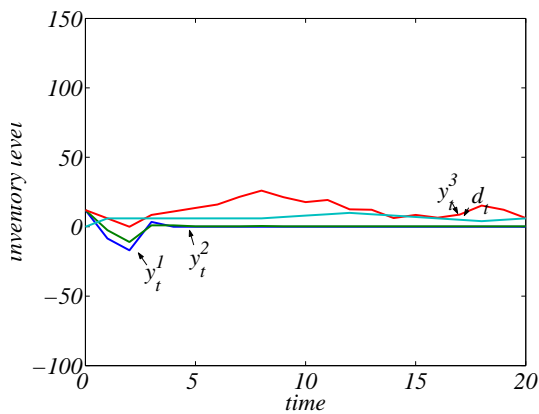
**Figure 7** Inventory behavior - ILV



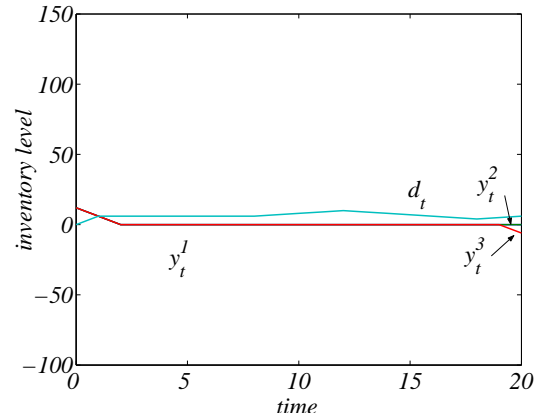
**Figure 8** Inventory behavior - RC



**Figure 9** Inventory behavior - AARC



**Figure 10** Inventory behavior - GRC



**Figure 11** Inventory behavior - Utopian

In the utopian solution, the ordered amount of each echelon in any given time is exactly the amount needed so that the inventory level will be zero, and so no holding or backlogging cost is paid. But even in this optimal method we can see "boundary behavior" in the beginning and end of the horizon. In the beginning, because the initial inventory level does not equal zero, it takes time for the system to stabilize and reach the optimal inventory level (zero). At the end of the horizon, the third echelon displays a negative inventory level. The reason is that if on the previous



time unit, the echelon send an order to its predecessor, echelon 2, it would incur a cost  $c_{20}^3 = 2$  for each unit ordered, consequently the second echelon could have chosen not to order and pay for the backlogging ( $p_{20}^2 = 3$  for each unit) or order from the first echelon (which costs  $c_{20}^2 = 2$  for each unit) and so on for echelon 1. Whatever the order of echelon 2 would have been, it would cost more than echelon 3 not ordering from echelon 2 and paying the backlogging cost. We also see similar "boundary behavior" in the robust methods.

In conclusion the robust methods dampen the Bullwhip effect and have the inventory behaving qualitatively similar to that of the utopian solution.

### 3.2. Cost Results

In this section we will refer to an example which deals with a step-type demand: the demand starts at a constant value and after some time assumes a new, higher value, and remains at this new level for the rest of the planning horizon. Saab and Corrêa (2005), use this type of demand, and according to Kirkwood (1998) such demand triggers the system in all its resonance frequencies, and therefore is very useful to characterize the system's behavior over time. We chose a demand that averages at 100 units for the first 10 periods and changes to an average of 150 units for the last 10 periods. We assume that the actual demand fluctuates within 10% around this averages.

We refer again to a three echelon supply chain ( $m=3$ ) and a horizon of  $n=20$ . Since it is known that longer lead times amplify the Bullwhip effect, and in order to accentuate this phenomenon, we chose  $L(j)=2$  and  $M(j)=I(j)=0$  for all  $j$ .

We also chose costs that do not make the problem trivial, such as  $c + h > p$  (otherwise the best policy will be not to order anything and pay the backlogging cost). Furthermore, we didn't impose a limit on the order size or the inventory/shortage level. A summary of all the example parameters is given in table 3.

**Table 3** step demand - data

Data type	Notation	Value
General	$n$	20
	$m$	3
Delays (for all $j$ )	$L(j)$	2
	$M(j)$	0
	$I(j)$	0
Costs (for all $j$ and $t$ )	$c_j^t$	2
	$h_j^t$	1
	$p_j^t$	3
Normal ranges for uncertainty sets (for all $j$ and $t$ )	$D_t$	$\begin{cases} [90, 110] & t \leq 10 \\ [135, 165] & t > 10 \end{cases}$
	$Z^j$	[180, 220]

We have used 4 different demand distributions in our simulations, which are given in table 4. Two of these demand distributions follow our assumption of a 10% fluctuation and the other two demand distributions have a 20% fluctuation around the average values.

We can see in table 5 that as in previous cases the average cost and deviation ratio tends to decrease as we move from the RC to the GRC. The methods are comparatively close to the utopian solution, and for both the AARC and GRC the cost is, on average, just 15% higher.

We also examine the empirical cdf of the deviation ratio -R for the various methods. The results are displayed in figures 12-15. In this figures, the X axis indicates R's value and the Y axis indicates the probability of a result being lower or equal to that given deviation value, i.e. the percentage of

**Table 4** Step demand - input distributions

Distribution	Input number	Demand ( $D_t$ )			Initial inventory ( $Z^j$ )	
Uniform		LB	UB	Relevance	LB	UB
	1(a)	90 135	110 165	$t \leq 10$ $t > 10$	180	220
	1(b)	80 120	120 180	$t \leq 10$ $t > 10$	160	240
Normal		Mean	Std	Relevance	Mean	Std
	2(a)	100 150	$3\frac{1}{3}$ 2.5	$t \leq 10$ $t > 10$	200	$6\frac{2}{3}$
	2(b)	100 150	$6\frac{2}{3}$ 5	$t \leq 10$ $t > 10$	200	$13\frac{1}{3}$

**Table 5** Step type demand - method comparison - average of the cost and R

Input	Measure	Average		
		RC	AARC	GRC
1(a)	Cost	14,910	14,330	14,330
	R	0.19	0.14	0.14
1(b)	Cost	15,598	14,386	14,386
	R	0.24	0.15	0.15
2(a)	Cost	14,701	14,271	14,113
	R	0.18	0.14	0.13
2(b)	Cost	15,112	14,335	14,229
	R	0.21	0.15	0.14

replications which resulted in R values that were lower or equal to the given value. The higher the graph is the better the method is (since it gives higher probability to results closer to the utopian solution). For most of these figures AARC and GRC are almost identical and exhibit low R values of no more than 0.3. For the RC which is the worst still exhibits R values of less than 0.5. We also notice that the RC the graphs tend to get less steep as we move to wider input distributions, so that the results tend to get further from the utopian solution. The other methods do not appear to exhibit this tendency and remain generally stable.

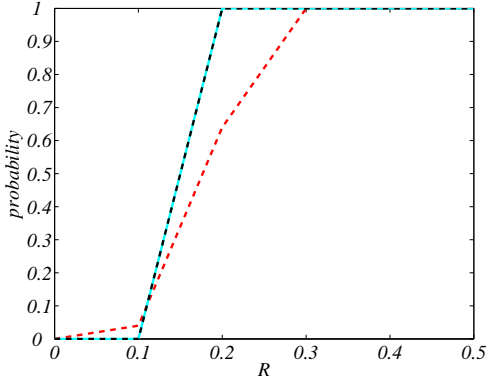


Figure 12 R's cdf - Input 1(a)

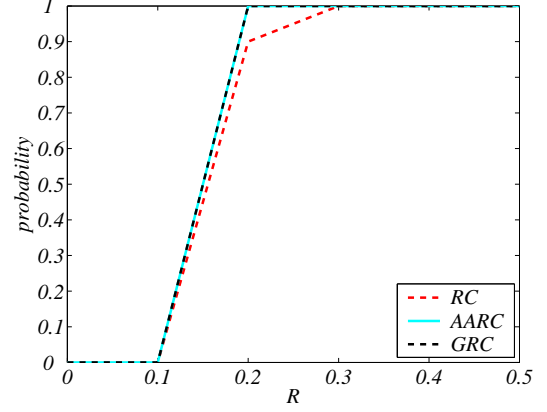


Figure 13 R's cdf - Input 2(a)

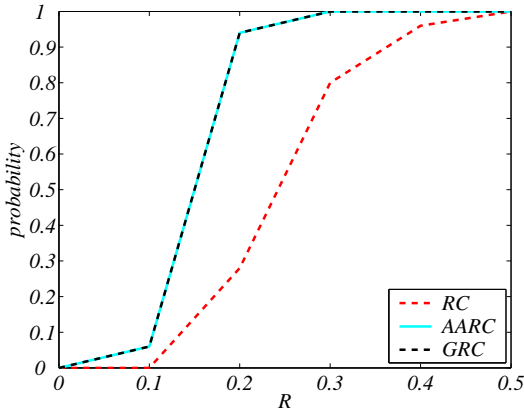


Figure 14 R's cdf - Input 1(b)

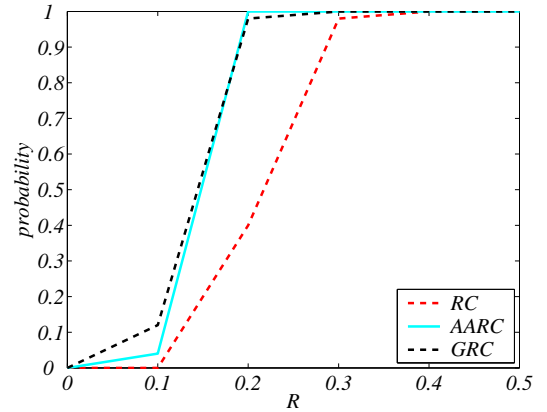


Figure 15 R's cdf - Input 2(b)

### 3.3. The Sensitivity Parameter $\alpha$

In this section we study the GRC method and its effect on the optimal cost. The main feature of the GRC that distinguishes it from the regular AARC, is the possibility to control the behavior of the system outside the normal range of the uncertain parameters (demands) by tuning the upper bound of the  $\alpha$ . Note that the AARC corresponds to the particular choice of  $\alpha = \infty$ .

We can examine the effect that the values of the upper limit of the  $\alpha$  vector have on the quality of the solution. Although the theoretical solution will be worse as we limit  $\alpha$  more and more, it is still a "minmax" solution, so the actual solutions can be much more optimistic.

Since the  $\alpha$  vector has many components we chose to limit ourselves to those  $\alpha$ -s associated with constraints which deal with the nonnegativity of  $x$  and  $w$ . The reason is that these constraints determine the coefficients in the control strategy ( $\eta_{\tau, j'}^{x, t, j}$ ,  $\eta_{\tau}^{w, t, j}$  and  $\tilde{\eta}_{j'}^{w, t, j}$ ). We examined 4 levels for upper bounds of the  $\alpha$  vector:  $\infty = \bar{\alpha}_0 \geq \bar{\alpha}_1 \geq \bar{\alpha}_2 \geq \bar{\alpha}_3 \geq \bar{\alpha}_4$ , (the vectors are ordered by the standard product order), and the bounds were chosen by trial and error and are different for each normal range assumed. Note that GRC with  $\bar{\alpha}_0 = \infty$  is equivalent to AARC. We studied the influence of each  $\alpha$ -value on the cdf of R (the deviation ratio). To explore this we use again the example shown in section 3.1 and simulate input taken from table 6. For each input distribution we simulated 50 realization to build the empirical cdf function for R.

Figures 16-21 depict the results of these realizations.

It can be seen quite clearly, that for *all* the input distributions that were examined, the results are essentially consistent. The AARC is better than the RC. As we restrict the  $\alpha$ -s more and more we expect worse results because the problem is more constrained. Indeed, the *robust guaranteed*

**Table 6** Test case - input distributions

Distribution	Input number	Demand ( $D_t$ )		Initial inventory ( $Z^j$ )	
		LB	UB	LB	UB
Uniform		4	10	10	14
	3(a)	3	11	9	15
	3(b)	2	12	8	16
Normal		Mean	Std	Mean	Std
	4(a)	7	1	12	2/3
	4(b)	7	1.5	12	1
	4(c)	7	2	12	2

*value* of the objective function does get bigger (1410 for both AARC and GRC with  $\bar{\alpha}_1$ , 1628 for GRC with  $\bar{\alpha}_2$ , 2055.5 for GRC with  $\bar{\alpha}_3$  and 2137 for GRC with  $\bar{\alpha}_4$ ). However, the simulation results (reflecting *average* performance) are quite surprising; we get better results using the GRC with  $\bar{\alpha}_1$  and  $\bar{\alpha}_2$  than using  $\bar{\alpha}_0$  (which is the AARC). We also expect the RC to be a lower bound on the results, because any restriction on  $\alpha$  still gives us more degrees of freedom than in the RC. But we find that, in general, using  $\bar{\alpha}_4$  and  $\bar{\alpha}_3$  generates worse results than those of the RC method. The explanation lies in the way the solution is obtained: The strategy generated by using these  $\alpha$ -s guarantees a better minmax objective function value than the RC method (objective value of 2055.5 and 2137 compared to 2137 in the RC solution) and so protects us better against worst case, but since the order strategy itself is different and depends on demand values the average performance is worsened.

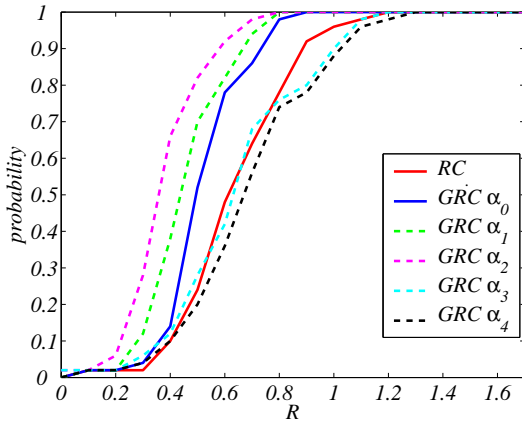
We can also see that for  $\bar{\alpha}_2$  we get the best results, and in particular, better results than the original AARC and RC solutions. This result is not trivial, because it indicates that although the theoretical robust objective function values, given by the solution of the GRC problem, become larger as we constrain  $\alpha$  more and more, the actual solutions, represented by the cdf, can be better with some upper limit on  $\alpha$  than with non. We chose the solution of the GRC with  $\bar{\alpha}_2$  to be represented as the GRC solution for the test case in the previous chapter, in order to show the potential of the GRC to improve our control.

We also should remember that the GRC was developed to protect us against demand variations that are outside the assumed uncertainty box. Therefore the 'b' and 'c' demand inputs taken from both normal and uniform distribution, are more adequate to this comparison than the 'a' inputs. We can see that even in these cases the same observations are valid.

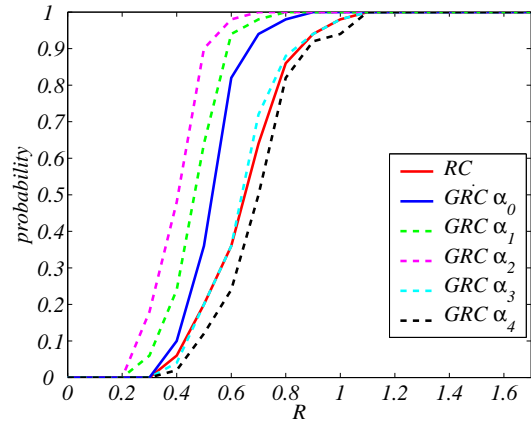
It is also interesting to note that although the best  $\alpha$  vector for each example is different, the orders of the GRC with constrained  $\alpha$  is always the same, and among them the  $\alpha_2$  GRC, which is moderately constrained, has the best results. This leads us to the conclusion that there is a balance between constraining the  $\alpha$  vector in the GRC formulation enough to cause it to converge to a low cost order strategy, and constraining it too much, depriving it the flexibility it needs to achieve good results.

### 3.4. The Normal Range

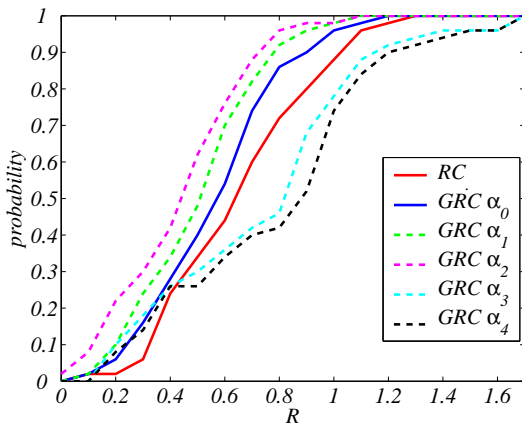
In the previous section we discussed how changing the  $\alpha$  impacts the results of our model. But the decision on the value of  $\alpha$  is dependent on the assumed normal range. Reducing the size of the normal range on one hand is less conservative, since we have to protect the system against lower uncertainty levels, but forces us to restrict  $\alpha$  so that under a real, wider demand range our control strategy will still yield feasible or close to feasible results.



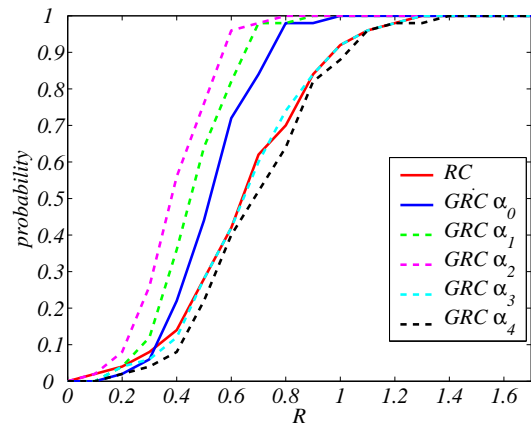
**Figure 16** comparing R's cdf between  $\alpha$  values - Input 3(a)



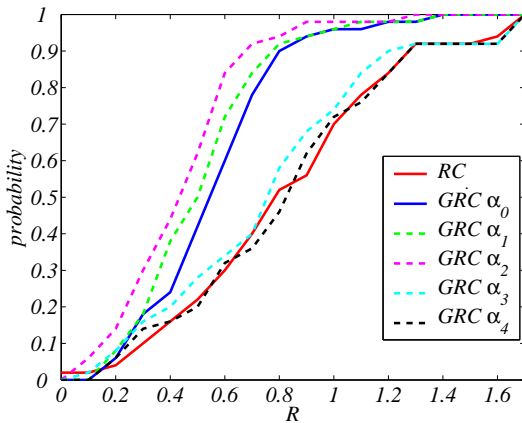
**Figure 17** comparing R's cdf between  $\alpha$  values - Input 4(a)



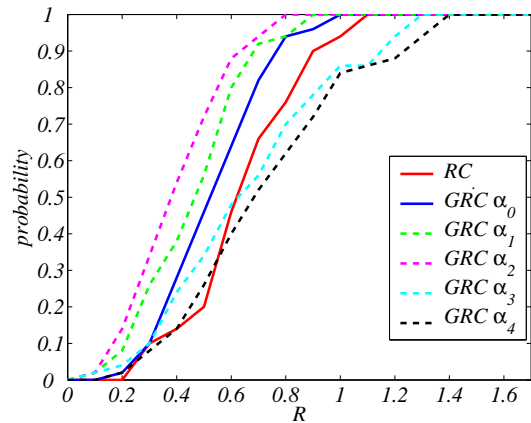
**Figure 18** comparing R's cdf between  $\alpha$  values - Input 3(b)



**Figure 19** comparing R's cdf between  $\alpha$  values - Input 4(b)



**Figure 20** comparing R's cdf between  $\alpha$  values - Input 3(c)



**Figure 21** comparing R's cdf between  $\alpha$  values - Input 4(c)

To demonstrate this, and in order to show the advantage of the GRC over the AARC method in controlling unexpected demand fluctuation, we study the following example, with parameters specified in table 2. In particular we are interested to observe how the change in the assumed normal range impacts the cost.

For the comparison we used the demand distribution support of the assumed normal range in table 2, and solved the GRC for three different normal ranges; the first has the same width as the support of the actual simulated demand distribution ( $[4,10]$ ), while the other two have a narrower width. The cost results are given in table 7.

**Table 7** Cost for different normal range values

Assumed normal range	AARC		GRC	
	Robust cost	Sampling worst case	Robust cost	Sampling worst case
$[4,10]$	1410	1240	1628	1124
$[5,9]$	1149	1100	1311	1068
$[6,8]$	944	1071	1044	1178

The GRC used for this example is the one with the upper bound on vector  $\alpha$  which incurs the lowest cost (found by trial and error). The robust cost given in the table, is the guaranteed objective function value given by the optimization of the AARC and GRC formulations and serves as a theoretical worst case value. The sampling worst case is the highest cost among the 50 replication, and serves as an estimation of the actual worst case value.

As expected, both the AARC and GRC robust costs decrease as we reduce the normal range, since the problem becomes less constrained. The actual worst cost, however, exhibits a different behavior, and for the smallest normal range is higher than the robust cost. This can be understood quite easily by the robust cost definition, which is the guaranteed value of the cost *within* the normal range, and so it does not predict the behavior of the cost if the demand exceeds this range. As we have seen before the AARC's robust cost values are lower than those of the GRC since the problem is less constrained but the actual worst case costs are not necessarily better. For normal range  $[5,9]$  both the GRC robust cost and its sampling worst case value are lower than that of the AARC robust cost for the normal range  $[4,10]$ , which corresponds to the actual demand values. This means that using a smaller normal range and protecting against infeasibility outside this range, through the GRC methodology, can provide us with better results than using the original AARC methodology. Using even a tighter normal range,  $[6,8]$ , still provides us with a better theoretical robust cost compared to the original AARC solution but generates a higher actual cost than the GRC solution for the wider,  $[5,9]$ , normal range. This leads us to believe that there is an optimal normal range which we should use in our solution, since it would yield the best results. The selection of this range is very problem dependent.

One can claim that we have neglected the AARC solution for the the normal ranges which are tighter than the actual demand and generated both lower robust cost and, in some instance, a lower actual cost than the corresponding GRC solutions. Our justification lies in the fact that the AARC solution does not protect us against the solution's behavior outside the normal range, hence using a normal range which is smaller might cause a highly infeasible solution. To test this claim we compared the infeasibility of the different solutions by counting the number of infeasible orders (smaller than 0) in each replication. Note that when a negative order occurs, we replace it by zero.

Table 8 displays the percentage of replications that had at least one infeasible order and the percentage that had more than one infeasible order.

We can see that in all replications the GRC does not have more than one infeasible order in all the assumed normal ranges, while more than 20% of the AARC replications have more than one infeasible order (some have up to 6 infeasible orders).

**Table 8** Percent of replications with infeasible orders

Assumed normal range	AARC		GRC	
	At least one infeasible order	More than one infeasible order	At least one infeasible order	More than one infeasible order
[4,10]	0%	0%	0%	0%
[5,9]	24%	22%	28%	0%
[6,8]	24%	24%	0%	0%

### 3.5. Information Sharing and Centralized Management

The supply chain problem we discussed so far assumes both information sharing between the various echelons and centralized inventory management. The information sharing is manifested in the fact that each echelon knows the external demand or can conclude it from the other echelons' orders. Centralized management, in this context, means that we optimize the system as a whole with the purpose of minimizing the common cost of the entire supply chain. Both of these assumptions, which together are the basis for the popular Vendor Managed Inventory (VMI) strategy, are very important in getting good overall results. Obviously, local optimum, and lack of information will lead to a distortion in the order strategy. This issue has been previously dealt with extensively in Simchi-Levi et al. (2000). Our methodology provides a qualitative way to evaluate the savings that can be obtained by using this strategy, even for complex problems of multi echelon multi period inventory control.

In order to test the hypothesis, that the existence of accurate information and centralized management is crucial for achieving efficient inventory control we tried to see what would be the implications on the cost if each echelon uses its own cost optimization, relying only on the order data given by its successor.

In order to see the effect of the information gap and the lack of centralized management on the total cost, we computed the total cost and the deviation ratio  $R$  and compared three methods:

1. The AARC method
2. The "local AARC theoretical" method - which is the AARC method applied separately for each echelon and the demand bounds used for the optimization rely on the theoretical bounds of the ordered amount given by the successor's strategy. It will be denoted by LAARCT.
3. The "local AARC actual" method - which is the AARC method applied separately for each echelon and the demand bounds used for the methods rely on the actual minimal and maximal orders given by the successor's strategy. It will be denoted by LAARCA.

For the comparison we used the randomized inputs as in the test case example described in section 6. The results are given in table 9. For the Local methods we added, in brackets, the corresponding value of the deviation ratio of the local method with respect to the original global method, we will denote this deviation ratio as  $R^G$ . For example in table 9 the numbers in brackets in the LAARCT column, denote the *average* deviation ratio of the LAARCT with respect to the AARC method for each input (this deviation ratio is calculated by  $R^G(LAARCT)_k = (\frac{LAARCT_k}{AARC_k} - 1)$  for each realization  $k$ ).

Some interesting conclusions can be derived from this table:

- As expected, the local methods give us, on average, worse results than the global methods. This fact is not surprising, and fits the extensive literature that stresses the importance of information sharing and centralized management.
- While the average deviation of the LAARCA method from the original AARC method increases with the width of the input range (from 1(a) and 2(a) to 1(c) and 2(c) respectively) the average deviation of the LAARCT from the AARC method is about 12% for all the inputs. This means that although the theoretical bounds is smaller than the actual one, the results achieved by

**Table 9** Local vs. Global Methods - average cost and R

Input	Measure	Average		
		AARC	LAARCA	LAARCT
3(a)	cost	1183	1308	1323
	R	0.62	0.79 (0.11)	0.81 (0.12)
3(b)	cost	1194	1412	1336
	R	0.65	0.95 (0.18)	0.84 (0.12)
3(c)	cost	1202	1544	1367
	R	0.65	1.12 (0.28)	0.88 (0.14)
4(a)	cost	1180	1215	1323
	R	0.64	0.69 (0.03)	0.84 (0.12)
4(b)	cost	1176	1332	1315
	R	0.62	0.84 (0.13)	0.81 (0.12)
4(c)	cost	1181	1426	1327
	R	0.63	0.97 (0.21)	0.83 (0.12)

using this bound are better as the actual input range gets larger. It also indicates, as we stated in the previous section, that **using a smaller uncertainty set than is actually expected may give on average better results than using the actual expected range.**

These differences between the Local and the Global methods show us that information sharing and centralized management can extensively impact the results. However it is still not clear what portion of the deviation is caused by the information gap and what portion by the lack of centralized management.

In order to understand these results better we turn to analyze the change in each of the echelons' costs (sum of their order and inventory cost) and see if these changes give us a better perspective of the problem. In tables 10-12 we can see the average of the cost for both the local and global methods and the  $R^G$  measure for each echelon.

**Table 10** Local vs. Global Methods - average retailer cost and  $R^G$ 

Input	Measure	Retailer (Echelon3)		
		AARC	LAARCA	LAARCT
3(a)	cost	570	407	407
	$R^G$		-0.29	-0.29
3(b)	cost	574	414	414
	$R^G$		-0.28	-0.28
3(c)	cost	584	426	426
	$R^G$		-0.27	-0.27
4(a)	cost	563	406	406
	$R^G$		-0.28	-0.28
4(b)	cost	559	405	405
	$R^G$		-0.28	-0.28
4(c)	cost	559	408	408
	$R^G$		-0.27	-0.27



**Table 11** Local vs. Global Methods - average wholesale cost and  $R^G$ 

Input	Measure	Wholesale (Echelon2)		
		AARC	LAARCA	LAARCT
3(a)	cost	291	428	433
	$R^G$		0.47	0.49
3(b)	cost	299	470	434
	$R^G$		0.57	0.45
3(c)	cost	296	523	444
	$R^G$		0.76	0.5
4(a)	cost	296	391	433
	$R^G$		0.32	0.47
4(b)	cost	298	449	431
	$R^G$		0.51	0.45
4(c)	cost	298	485	434
	$R^G$		0.63	0.46

**Table 12** Local vs. Global Methods - average manufacturer cost and  $R^G$ 

Input	Measure	Manufacturer (Echelon1)		
		AARC	LAARCA	LAARCT
3(a)	cost	271	472	481
	$R^G$		0.75	0.78
3(b)	cost	322	528	487
	$R^G$		0.64	0.52
3(c)	cost	2321	595	496
	$R^G$		0.85	0.54
4(a)	cost	321	418	481
	$R^G$		0.3	0.5
4(b)	cost	319	478	478
	$R^G$		0.5	0.5
4(c)	cost	319	533	483
	$R^G$		0.67	0.51

Some observations are due:

- As to be expected, the local methods improve the cost of the retailer since he does not have to take into account the consequences that his orders may have on the higher echelons. The local AARC methods exhibit cost as low as 70% of the original AARC's cost.
- The LAARCT and LAARCA are identical in the retailer level, because they both assume the same demand.
- At wholesale level we see a **higher** cost in the local methods than the global ones. This is due to the lack of information on the retailer's orders. The local AARC methods have higher costs than the global AARC by 30%-75%.
- At manufacturer level we see the same trends as in the wholesale level. The local AARC methods are higher than the global AARC by 30%-85%.
- In the wholesale and manufacturer level the LAARCT is more stable due to its reliance on theoretical orders.

The VMI method is better than local inventory management, provided the added value to the wholesale and manufacturer compensates the increase in the retailer's cost, and the added value is shared by all the echelons. In our example we can see that for input 3(c) if the retailer uses its

local method the cost incurred is 426 vs. 584 in the global AARC. At the same time the wholesaler and manufacturer together save 500 cost units. If they compensate the retailer by 158 units, this leaves them with total savings of 342 which they can share among the echelons according to some predetermined contract. This issue is addressed in Cachon and Lariviere (2005), which compare several types of revenue sharing contracts to encourage coordination and information sharing.

## 4. Summary and future research

### 4.1. Summary

In this work we develop an instrument for supply chains control. We examine serial supply chains, which are the building blocks of every type of supply chain. The objective of such a control is first and foremost reducing the system's cost, even under uncertain demand, in order to make it more profitable. An additional, byproduct, of this objective would be to stabilize the system under such unstable (or uncertain) external demand.

We adapt a general and applicable control solution for supply chains, trying to optimize it from an economical point of view, implementing principles corresponding to the VMI framework and avoiding forecasting problems by dealing with robust and implicit forecasting. For this purpose, we apply the GRC methodology, to supply chain control. The GRC methodology is a robust way to deal with uncertainty without using prediction methods that might influence system stability. It is also less strict than the AARC method, as it allows us to use a less conservative approach on the less likely values of the uncertainty set. We pay for this flexibility with problem dimensional limitations, as a problem of 3 echelons and 20 time periods in the planning horizon contains approximately 40,000 constraints and 25,000 variables.

We compare the GRC methodology with the RC and AARC formulations. We also use the utopian solution, an unrealistic solution which finds the optimal control when all the demand values are known in advance, as a reference point for the different methods. We show that the robust methods dampen the Bullwhip effect and obtain good results with respect to the utopian solution.

We further investigate the GRC method itself, especially its behavior outside the original (normal) uncertainty set. This behavior is a function of the  $\alpha$  parameter and the assumed normal range. We discovered that there is an "optimal"  $\alpha$  vector that will yield the best practical results for any given input (depending on the shape and width of its distribution). We can also conclude that the cost function is a non-monotonic function of  $\alpha$ . The choice of the value for  $\alpha$  and the resulting performance of the system is highly dependent on the non-trivial choice of the normal uncertainty set. We also find that choosing a smaller normal set for the uncertainty in the GRC methodology can actually improve both theoretical and practical results and still protect us against solution infeasibility.

Lastly we discuss the impact of information and management structure assumptions, showing that both vertical (among echelons) information sharing and horizontal (time dependent) information updating, has a substantial impact on the cost. We demonstrate that using VMI and revenue management concepts can improve the solution.

### 4.2. Future research directions

Our work has laid down the foundation for using the GRC control methodology for supply chain control with demand uncertainty. The scope of our research can be expanded in future research to deal with additional issues concerning the supply chain control problem. Next we discuss several of the directions we foresee for this methodology being developed in future research.

One of the more straightforward extensions of the model is to include a wider range of supply chains, such as distribution supply chains. The dynamics of such a system changes in a very simple way and is represented in equation (27).

$$\begin{aligned}
y_t^{j_l} &= y_{t-1}^{j_l} + x_{t-T^L(j_l)}^j - \sum_{(j+1)_k \in B(j_l)} x_{t-T^M(j_l)}^{(j+1)_k} \quad \forall j \in \{1, \dots, m-1\} \quad \forall l \in 1, \dots, r_j \\
y_t^{m_l} &= y_{t-1}^{m_l} + x_{t-T^L(m_l)}^m - d_{t-T^M(m)}^{m_l} \quad \forall l \in 1, \dots, r_m \\
y_0^{j_l} &= z^{j_l} \quad \forall j \in \{1, \dots, m\} \quad \forall l \in 1, \dots, r_j
\end{aligned} \tag{27}$$

We assume that each echelon  $j$  has  $r_j$  vendors and each vendor  $j_l$  of echelon  $j$  supplies to the vendors in echelon  $j+1$  that belong to group  $B(j_l)$  (their successors). We denote  $x_t^{j_l}$  as vendor  $j_l$ 's orders at time  $t$  and  $y_t^{j_l}$  as his inventory level at time  $t$ . There is an external demand for each vendor in the downstream echelon, vendor  $m_l$  in echelon  $m$ , which is denoted by  $d_t^{m_l}$ . This dynamic behavior replaces the serial supply chain dynamics which is given in equation (1) to that of the distribution supply chain.

Since the dynamics of the system is such that all linear attributes are maintained and it still corresponds with both control system dynamics and GRC tractability requirements, we can apply the same process to develop a GRC model for this type of supply chain.

Furthermore, our model is simple because we assume that the state variables (in our case, the inventory status) are the same as observed outputs. Meaning that we always know the exact inventory. The original control model, however, has a linear relationship between the state variable  $\mathcal{X}$  and output  $\mathcal{Y}$  so that  $\mathcal{Y}_t = C_t \mathcal{X}_t$ . So, we can also assume that we do not see the exact inventory state but some distortion of this state which is due to inaccurate reports represented by matrices  $C_t$ . Extending the scope of the model to include this distortion is instrumental in understanding what is the effect of different kinds of errors on the system performance on one hand, and the effectiveness of the GRC methodology on controlling these errors, on the other hand.

There are many other issues that were raised in this work that can be further explored. One of these issues might be understanding the relationship and tradeoff between the assumed demand normal range and the limits on the GRC parameter  $\alpha$  in order to calibrate these parameters and consequently achieve better control. Another research direction is to investigate the length of the planning horizon we need to look at in order to balance the problem dimensions with solution effectiveness and devise a method of finding that length.

## References

- Arrow, K.J., Karlin S., Scarf H. 1958. *Studies in The Mathematical Theory of Inventory and Production*. Stanford University Press, Stanford, California.
- Ben-Tal, A., S. Boyd, A. Nemirovski. 2005. "extending the scope of robust optimization : Comprehensive robust counterparts of uncertain problems". [http://www.optimization-online.org/DB\\_HTML/2005/05/1136.html](http://www.optimization-online.org/DB_HTML/2005/05/1136.html).
- Ben-Tal, A., A. Goryashko, E. Guslitzer, A. Nemirovski. 2004. "adjustable robust solutions of uncertain linear programs". *Mathematical Programming* **99** 351–376.
- Ben-Tal, A., A. Nemirovski. 2002. "a robust optimization - methodology and application". *Mathematical Programming (Series B)* **92** 453–480.
- Blanchard, O.J. 1983. "the production and inventory behavior of the american automobile industry". *Journal of Political Economy* **91**.
- Cachon, G.P., M.A. Lariviere. 2005. "supply chain coordination with revenue-sharing contracts: Strengths and limitations". *Management Science* **51**(1) 30–44.
- Cachon, G.P., R. Taylor, G.M. Schmidt. 2005. "in search of the bullwhip effect". <http://opim.wharton.upenn.edu/~cachon/pdf/bwv2.pdf>.
- Chen, F., Z. Drezner, J. Ryan, D. Simchi-Levi. 2000. "quantifying the bullwhip effect in a simple supply chain: The impact of forecasting, lead times and information". *Management Science* **46** 436–43.
- Disney, S.M., D.R. Towill, W. Van de Velde. 2004. "variance amplification and the golden ratio in production and inventory control". *International Journal of Production economics* **90** 295–309.
- Forrester, J.W. 1973. *Industrial Dynamics*. 8th ed. MIT Press, Cambridge, Massachusetts.
- Hammond, J. 1994. Barilla spa (a). Harvard Business School case # 9-694-046.
- Holt, C.C., F. Modigliani, J.P. Shelton. 1968. "the transmission of demand fluctuations through distribution and production systems: The tv-set industry". *Canadian Journal of Economics* **14** 718–39.
- Kirkwood, C.W. 1998. *System Dynamics Methods: A Quick Introduction*. Arizona State University.
- Lee, H., V. Padmanabhan, S. Whang. 1997a. The bullwhip effect in supply chains. *MIT Sloan Management Review* **38** 93–102.
- Lee, H., V. Padmanabhan, S. Whang. 1997b. "information distortion in a supply chain:the bullwhip effect". *Management Science* **43** 546–58.
- Lee, H., V. Padmanabhan, S. Whang. 2004. "comments on information distortion in a supply chain: The bullwhip effect". *Management Science* **50** 1887–93.
- Lin, P.H., D.S.H. Wong, S.S. Jang, S.S. Shieh, J.Z. Chu. 2004. "controller design and reduction of bullwhip for model supply chain system using z-transform analysis". *Journal of Process Control* 487–499.
- Love, S. 1979. *Inventory Control*. McGraw-Hill.
- Saab, J., E. Corrêa. 2005. "bullwhip effect reduction in supply chain management: one size fits all?". *Int. J. Logistics Systems and Management* **1**(2/3) 211–226.
- Simchi-Levi, D., P. Kaminski, E. Simchi-Levi. 2000. *Designing and Managing the Supply Chain*. McGraw-Hill.
- Sterman, J.B. 1989. "modeling managerial behavior: Misperceptions of feedback in a dynamic decision making experiment?". *Management Science* **35** 321–39.
- Sterman, J.B. 2000. *Dynamics: Systems Thinking and Modeling for a Complex World*. Irwin-McGraw-Hill, New York.
- Terwiesch, C., Ren J., T.H. Ho, Cohen M. 2005. "forecast sharing in the semiconductor equipment supply chain". *Management Science* **51** 208–20.
- Whitin, T.M. 1957. *The Theory of Inventory Management*. 2nd ed. Princeton University Press, Princeton, New Jersey.
- Zhang, X. 2005. "delayed demand information and dampened bullwhip effect". *Operations Research letters* **33** 289–294.