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NONUNIQUENESS FOR A PARABOLIC SPDE WITH $\frac{3}{4}-\varepsilon$-HÖLDER DIFFUSION COEFFICIENTS

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Motivated by Girsanov's nonuniqueness examples for SDEs, we prove nonuniqueness for the parabolic stochastic partial differential equation (SPDE)

$$\frac{\partial u}{\partial t} = \frac{\Delta}{2} u(t,x) + |u(t,x)|^{\gamma} \dot{W}(t,x), \qquad u(0,x) = 0.$$

Here $\dot{W}$ is a space–time white noise on $\mathbb{R}_+ \times \mathbb{R}$. More precisely, we show the above stochastic PDE has a nonzero solution for $0 < \gamma < 3/4$. Since $u(t,x) = 0$ solves the equation, it follows that solutions are neither unique in law nor pathwise unique. An analogue of Yamada–Watanabe's famous theorem for SDEs was recently shown in Mytnik and Perkins [*Probab. Theory Related Fields* **149** (2011) 1–96] for SPDE's by establishing pathwise uniqueness of solutions to

$$\frac{\partial u}{\partial t} = \frac{\Delta}{2} u(t,x) + \sigma(u(t,x)) \dot{W}(t,x)$$

if σ is Hölder continuous of index $\gamma > 3/4$. Hence our examples show this result is essentially sharp. The situation for the above class of parabolic SPDE's is therefore similar to their finite dimensional counterparts, but with the index 3/4 in place of 1/2. The case $\gamma = 1/2$ of the first equation above is particularly interesting as it arises as the scaling limit of the signed mass for a system of annihilating critical branching random walks.

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1. Introduction. This work concerns uniqueness theory for parabolic semilinear stochastic partial differential equations (SPDE) of the form

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{\Delta}{2}u(t,x) + \sigma(x,u(t,x))\dot{W}(t,x), \\ &u(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

where $\dot{W}(t,x)$ is two-parameter white noise on $\mathbb{R}_+ \times \mathbb{R}$, and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is γ-Hölder continuous in u and also has at most linear growth at ∞ in u. See $(2.1)'$ in Shiga (1994) or (1.7) below for a precise definition of a solution. Weak existence of solutions in the appropriate function space is then standard; see, for example, Theorems 1.1 and 2.6 of Shiga (1994) or Theorem 1.1 of Mytnik and Perkins (2011). If $\gamma = 1$, then σ is Lipschitz in u, and pathwise uniqueness of solutions follows from standard fixed-point arguments; see Chapter 3 in Walsh (1986). A natural question is then:

If $\gamma < 1$, are solutions pathwise unique?

The motivation for this problem comes from a number of models arising from branching processes and population genetics for which $\gamma = 1/2$.

Next we give some examples. In the first three, we only consider nonnegative solutions, while in the fourth example we allow solutions to take negative values. If $E \subset \mathbb{R}$, we write $C(E)$ for the space of continuous functions on E with the topology of uniform convergence on compact sets.

EXAMPLE 1. If $\sigma(u) = \sqrt{u}$ and we assume $u \geq 0$, then a solution to (1.1) corresponds to the density $u(t,x)\,dx = X_t(dx)$, where X_t is the one-dimensional super-Brownian motion. The super-Brownian motion is a measure-valued process which arises as the rescaled limit of branching random walks; see Reimers (1989) and Konno and Shiga (1988). More precisely, assume that particles occupy sites in $\mathbb{Z}/\sqrt{N}$. With Poisson rate $N/2$, each particle produces offspring at a randomly chosen nearest neighbor site. Finally, particles die at rate $N/2$. For $x \in \mathbb{Z}/\sqrt{N}$ and $t \geq 0$, set

$$U^N(t,x) = N^{-1/2} \times (\text{number of particles at } x \text{ at time } t).$$

If the initial "densities" converge in the appropriate state space, then U^N will converge weakly on the appropriate function space to the solution of (1.1), with σ as above; see Reimers (1989) for a proof of this result using nonstandard analysis. Furthermore, this solution is unique in law. Uniqueness in law is established by the well-known exponential duality between $u(t,x)$ and solutions $v(t,x)$ of the semilinear PDE

$$\frac{\partial v}{\partial t} = \frac{\Delta v}{2} - \frac{1}{2}v^2.$$

One of us [Mytnik (1998)] extended this exponential duality and hence proved uniqueness in law for $\sigma(u) = u^p, u \geq 0$ where $1/2 < p < 1$. The dual process is then a solution to an SPDE driven by a one-sided stable process. Pathwise uniqueness among nonnegative solutions remains unsolved for $0 < p \leq 3/4$; see below for $p > 3/4$.

EXAMPLE 2. If $\sigma(x, u) = \sqrt{g(x, u)u}, u \geq 0$, where g is smooth, bounded, and bounded away from 0, then any kind of uniqueness for solutions to (1.1) is unresolved except when g is constant. Such equations arise as weak limit points of the branching particle systems as in Example 1, but where the branching and death rates of a particle at x in population u^N is $Ng(x, u^N)/2$.

EXAMPLE 3. If $\sigma(x, u) = \sqrt{u(1-u)}$, $u \in [0, 1]$, then solutions to (1.1) are population densities for the stepping stone model on the line. That is, $u(t, x)$ is the proportion of a particular allele type at location x in a population undergoing Brownian migration and resampling between generations. For this model, uniqueness in law holds by a moment duality argument [see Shiga (1988)], and pathwise uniqueness remains unresolved.

EXAMPLE 4. In this example, we no longer require u to be nonnegative. Consider $\sigma(u) = \sqrt{|u|}$ for $u \in \mathbb{R}$; that is, consider the SPDE

$$\frac{\partial u}{\partial t}(t, x) = \frac{\Delta}{2} u(t, x) + \sqrt{|u(t, x)|}\dot{W}(t, x). \tag{1.2}$$

This equation arises as a weak limit of the signed particle density of two branching random walks, one with positive mass and one with negative mass, which annihilate each other upon collision. More precisely, consider two particle systems on $\mathbb{Z}/\sqrt{N}$, one with positive mass and the other with negative mass. Each particle independently produces offspring of the same sign at a randomly chosen nearest neighbor at rate $N/2$ and dies at rate $N/2$. The systems interact when particles collide, and then there is pairwise annihilation. Define $U^{N,\pm}(t, x)$ as in Example 1 where one considers separately the positive and negative masses. Extend these functions by linear interpolation to $x \in \mathbb{R}$. If $U^{N,\pm}(0, \cdot) \to u^{\pm}(0, \cdot)$ uniformly for some limiting cadlag (right-continuous with left limits) functions with compact support satisfying $u^+(0, \cdot)u^-(0, \cdot) \equiv 0$, then $\{(U^{N,+}, U^{N,-}) : N \in \mathbb{N}\}$ is tight in the Skorokhod space of cadlag $C(\mathbb{R})$-valued paths, where the latter space of continuous functions has the topology of uniform convergence on compact sets. Any weak limit point (u^+, u^-) will satisfy

$$\begin{aligned} &\frac{\partial u^{\pm}}{\partial t}(t, x) = \frac{\Delta}{2} u^{\pm}(t, x) + \sqrt{u^{\pm}(t, x)}\dot{W}_{\pm}(t, x) - \dot{K}_t, \\ &u^+(t, x)u^-(t, x) \equiv 0, \end{aligned} \tag{1.3}$$

where $\dot{W}_+$ and $\dot{W}_-$ are independent space–time white noises and K_t is a continuous nondecreasing process taking values in the space of finite measures on the line with the topology of weak convergence. The space–time measure $K(dt, dx)$ records the time and location of the killing resulting from the particle collisions. It is then easy to check that $u = u^+ - u^-$ satisfies (1.2). No results about uniqueness were known for this process. The above convergence was proved in an earlier draft of this article but we have not included it as the details are a bit lengthy, if routine. The convergence will only be used to help our intuition in what follows.

In general, pathwise uniqueness of solutions, that is, the fact that two solutions with the same white noise and initial condition must coincide a.s., implies the uniqueness of their laws; see, for example, Kurtz (2007). Quite different duality arguments give uniqueness in law in Examples 1 and 3, at least among nonnegative solutions. But this kind of duality argument is notoriously nonrobust, and the interest in pathwise uniqueness stems in part from the hope that such an approach would apply to a broader class of examples, including perhaps Examples 2 and 4.

It has long been hoped that pathwise uniqueness holds in (1.1) if σ is γ-Hölder continuous in the solution u for $\gamma \geq 1/2$, since Yamada and Watanabe (1971) showed the corresponding result holds for finite-dimensional stochastic differential equations (SDEs). They proved that if $\sigma_i : \mathbb{R} \to \mathbb{R}$ is Hölder continuous of index 1/2 and $b_i : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous, then solutions to

$$dX_t^i = \sigma_i(X_t^i)\, dB_t^i + b_i(X_t)\, dt, \qquad i = 1, \dots, d$$

are pathwise unique. Note that (1.1) has the same "diagonal form" as the above SDE albeit in infinitely many dimensions. It was Viot (1975) who first noted Yamada and Watanabe's proof extends to infinite dimensional equations such as (1.1) if the noise is white in time but has a bounded covariance kernel in the spatial variable. This proof breaks down for noise that is white in both time and space, since in the t variable, solutions are Hölder continuous of index $(1/4) - \varepsilon$ for all $\varepsilon > 0$, but not Hölder continuous of index 1/4. Hence, solutions are too rough in the time variable to be semimartingales. Nonetheless in Mytnik and Perkins (2011) a more involved extension of the Yamada–Watanabe argument was established which proved pathwise uniqueness in (1.1) if $\sigma(x, \cdot)$ is Hölder continuous of index $\gamma > 3/4$, uniformly in x.

This leads to the natural question of sharpness in this last result, that is:

(1.4) Does pathwise uniqueness fail in general for (1.1) if $\sigma(x, \cdot) = \sigma(\cdot)$ is γ-Hölder continuous for $\gamma \leq 3/4$, and in particular for $\gamma = 1/2$?

For the corresponding SDE, the Yamada–Watanabe result is shown to be essentially sharp by Girsanov's equation

$$X_t = \int_0^t |X_s|^\gamma \, dB_s \tag{1.5}$$

for which one solution is $X_t = 0$. If $\gamma < 1/2$, there are nonzero solutions to (1.5), and so solutions are neither pathwise unique nor unique in law; see Section V.26 in Rogers and Williams (1987). This suggests we consider the SPDE

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{\Delta}{2} u(t,x) + |u(t,x)|^\gamma \dot{W}(t,x), \\ u(0,x) &= 0. \end{aligned} \tag{1.6}$$

To state our main result we need some notation. A superscript k, respectively ∞, indicates that functions are in addition k times, respectively infinitely often, continuously differentiable. A subscript b, respectively c, indicates that they are also bounded (together with corresponding derivatives), respectively have compact support. Let $\langle f, g\rangle = \int_{\mathbb{R}} f(x) g(x)\, dx$ denote the L^2 inner product. Set

$$\|f\|_\lambda := \sup_{x\in\mathbb{R}} |f(x)| e^{\lambda |x|},$$

and define $C_{\mathrm{rap}} := \{f \in C(\mathbb{R}) : \|f\|_\lambda < \infty \text{ for any } \lambda > 0\}$, endowed with the topology induced by the norms $\|\cdot\|_\lambda$ for $\lambda > 0$. That is, $f_n \to f$ in C_{rap} if and only if $d(f, f_n) = \sum_{k=1}^\infty 2^{-k}(\|f - f_n\|_k \wedge 1) \to 0$ as $n \to \infty$. Then (C_{rap}, d) is a Polish space. The space C_{rap} is a commonly used state space for solutions to (1.1); see Shiga (1994).

We assume in (1.1) that $\dot{W}$ is a white noise on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\mathcal{F}_t$ satisfies the usual hypotheses. This means $W_t(\phi)$ is an $\mathcal{F}_t$-Brownian motion with variance $\|\phi\|_2^2 t$ for each $\phi \in L^2(\mathbb{R}, dx)$ and $W_t(\phi_1)$ and $W_t(\phi_2)$ are independent if $\langle \phi_1, \phi_2\rangle = 0$. A stochastic process $u : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ which is $\mathcal{F}_t$-previsible $\times$ Borel measurable will be called a solution to the SPDE (1.1) with initial condition $u_0 : \mathbb{R} \to \mathbb{R}$ if for each $\phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} \langle u_t, \phi\rangle &= \langle u_0, \phi\rangle + \int_0^t \left\langle u_s, \frac{\Delta}{2}\phi \right\rangle ds \\ &\quad + \int_0^t \int \sigma(x, u(s,x)) \phi(x) W(ds, dx) \qquad \text{for all } t \ge 0 \text{ a.s.} \end{aligned} \tag{1.7}$$

(The existence of all the integrals is of course part of the definition.) We often write u_t for $u(t, \cdot)$. We use the framework of Walsh (1986) to define stochastic integrals with respect to $W(ds, dx)$. For $u_0 \in C_{\mathrm{rap}}$, we say u is a

C_{rap}-valued solution if, in addition, $t \to u(t,\cdot)$ has continuous C_{rap}-valued paths for all ω.

Here is our main result which answers question (1.4), at least for $\gamma < 3/4$.

THEOREM 1.1. *If $0 < \gamma < 3/4$, there is a C_{rap}-valued solution $u(t,x)$ to (1.6) such that with positive probability, $u(t,x)$ is not identically zero. In particular, uniqueness in law and pathwise uniqueness fail for (1.6).*

This leaves open the state of affairs for $\gamma = 3/4$ where, based on analogy with the SDE, one would guess that uniqueness holds. Our theorem does, however, dampen the hope of handling many of the SPDE's in the above examples through a Yamada–Watanabe type theorem. It also shows that the SPDE in Example 4 does not specify a unique law.

A standard construction of a nonzero solution to Girsanov's SDE proceeds as follows. Start an "excursion" from $\pm\varepsilon$, run it until it hits 0, and then proceed to the next excursion, starting with the opposite sign. The process consisting of $\pm\varepsilon$ jumps will disappear as $\varepsilon \to 0$ due to the alternating signs. For $\gamma < 1/2$, a diffusion calculation shows that the rescaled return time of the diffusion is in the domain of attraction of a stable subordinator of index $(2(1-\gamma))^{-1} < 1$, and the limiting jumps will lead to nontrivial excursions in the scaling limit. With a bit of work one can do the same in (1.6) for $\gamma < 1/2$. That is, one can seed randomly chosen bits of mass of size $\pm\varepsilon$ and run the SPDE until it hits 0 and try again. Theorem 4 of Burdzy, Mueller and Perkins (2010) carries out this argument and gives Theorem 1.1 for $\gamma < 1/2$. Therefore, in the rest of this work we will assume

$$1/2 \le \gamma < 3/4. \tag{1.8}$$

When $\gamma \ge 1/2$ the above excursion argument breaks down as the time to construct a nontrivial excursion will explode. Instead we start excursions which overlap in time and deal with the potential spatial overlap of positive and negative excursions. As Example 4 suggests we will annihilate mass when the overlap occurs. Much of the challenge will be to show that this overlap can be quite small if $\gamma < 3/4$.

We now outline our strategy for constructing a nontrivial solution to (1.6). Let $M_F(E)$ denote the space of finite measures on the metric space E with the weak topology. We will also use $\mu(\phi)$ and $\langle \mu, \phi\rangle$ to denote integral of a function ϕ against a measure μ. Below we will construct $\eta_\varepsilon^+, \eta_\varepsilon^- \in M_F([0,1]^2)$, both of which converge to Lebesgue measure $dt\,dx$ on the unit square as $\varepsilon \downarrow 0$, and we will also construct nonnegative solutions $U^\varepsilon(t,x)$ and $V^\varepsilon(t,x)$ with 0 initial conditions to the equations

$$\frac{\partial U^\varepsilon}{\partial t}(t,x) = \dot\eta_\varepsilon^+(t,x) + \frac{\Delta}{2} U^\varepsilon(t,x) + U^\varepsilon(t,x)^\gamma \dot W^+(t,x) - \dot K_t^\varepsilon, \tag{1.9}$$

$$\frac{\partial V^\varepsilon}{\partial t}(t,x) = \dot\eta_\varepsilon^-(t,x) + \frac{\Delta}{2} V^\varepsilon(t,x) + V^\varepsilon(t,x)^\gamma \dot W^-(t,x) - \dot K_t^\varepsilon. \tag{1.10}$$

Here $\dot{W}^+$ and $\dot{W}^-$ are independent white noises, and $t \to K_t^\varepsilon$ is a nondecreasing $M_F(\mathbb{R})$-valued process. As suggested by (1.3), $K^\varepsilon(dt, dx)$ will record the locations of the pairwise annihilations resulting from the collisions between our two annihilating populations. This construction will lead to the condition

$$U^\varepsilon(t,\cdot)V^\varepsilon(t,\cdot) \equiv 0.$$

Note that $\eta_\varepsilon^\pm$ are immigration terms. We will always assume that $\varepsilon \in (0,1]$. If $\eta_\varepsilon = \eta_\varepsilon^+ - \eta_\varepsilon^-$, it is easy to check that $u_\varepsilon = U^\varepsilon - V^\varepsilon$ satisfies

$$\frac{\partial u_\varepsilon}{\partial t}(t,x) = \dot{\eta}_\varepsilon(t,x) + \frac{\Delta}{2}u_\varepsilon(t,x) + |u_\varepsilon(t,x)|^\gamma \dot{W}(t,x) \tag{1.11}$$

for an appropriately defined white noise $\dot{W}$. We will show that there exists a subsequence ε_k such that as $k \to \infty$, $u_{\varepsilon_k}(t,x)$ converges weakly in the Skorokhod space of C_{rap}-valued paths to a solution $u(t,x)$ of (1.6); see Proposition 2.2. U^ε is the positive part of u_ε, and so Theorem 1.1 will then follow easily from the following assertion:

Claim 1.2. *There exists $\delta > 0$ such that for all $\varepsilon \in (0,1]$,*

$$P\left(\sup_{t\in[0,1]} \int U^\varepsilon(t,x)\,dx > \delta\right) > \delta.$$

If $N_\varepsilon = \lfloor \varepsilon^{-1} \rfloor$ (the greatest integer less than ε^{-1}), the measure η_ε will be obtained by smearing out spatial mass using the time grid

$$\mathcal{G}_\varepsilon = \{k\varepsilon/2 : 1 \le k \le 2N_\varepsilon\}. \tag{1.12}$$

We further denote by $\mathcal{G}_\varepsilon^{\mathrm{odd}}$ the points of $\mathcal{G}_\varepsilon$ for which k is odd, where k is in the definition of $\mathcal{G}_\varepsilon$ above. We also define $\mathcal{G}_\varepsilon^{\mathrm{even}}$ to be those grid points for which k is even and let

$$J_\varepsilon^x(z) = \varepsilon^{1/2} J((x-z)\varepsilon^{-1/2}), \qquad x, z \in \mathbb{R}, \tag{1.13}$$

where J is a nonnegative even continuous function bounded by 1 with support in $[-1,1]$, and such that $\int_{\mathbb{R}} J(z)\,dz = 1$. Now let us enumerate points in $\mathcal{G}_\varepsilon^{\mathrm{odd}}$ and $\mathcal{G}_\varepsilon^{\mathrm{even}}$ as follows:

$$\{s_i, i \in \mathbb{N}_\varepsilon\} = \mathcal{G}_\varepsilon^{\mathrm{odd}}, \qquad \{t_i, i \in \mathbb{N}_\varepsilon\} = \mathcal{G}_\varepsilon^{\mathrm{even}},$$

where $s_i = (2i-1)\frac{\varepsilon}{2}$ and $t_i = 2i\frac{\varepsilon}{2}$ for $i \in \mathbb{N}_\varepsilon = \{1, \dots, N_\varepsilon\}$. Let $x_i, y_i, i = 1, 2, \dots,$ be a sequence of independent random variables distributed uniformly on $[0,1]$.

We define η_ε to be the signed measure

$$\eta_\varepsilon(A) = \left[\sum_{s_i\in\mathcal{G}_\varepsilon^{\rm odd}} \int J_\varepsilon^{x_i}(y) 1_A(s_i,y)\,dy - \sum_{t_i\in\mathcal{G}_\varepsilon^{\rm even}} \int J_\varepsilon^{y_i}(y) 1_A(t_i,y)\,dy\right]$$
$$\equiv \eta_\varepsilon^+(A) - \eta_\varepsilon^-(A).$$

It is easy to check that $\eta_\varepsilon^\pm$ are as claimed above.

To simplify the outline of our proof, we will take $\gamma = 1/2$ so that we can appeal to Example 4 for intuition. In later sections we do not make this restriction on γ. We can then decompose $U^\varepsilon = \sum_{i=1}^{N_\varepsilon} U^i$ into descendants of the ith immigrant at (s_i, x_i) (type i particles) and similarly write $V^\varepsilon = \sum_{j=1}^{N_\varepsilon} V^j$. We will suppress ε in the notation for clusters U^i and V^j. We can also keep track of the killed mass and, by adding these ghost particles back in, dominate U^ε by a super-Brownian motion $\bar{U}$ with immigration η_ε^+, and dominate the $\{U^i\}$ by independent super-Brownian motions $\{\bar{U}^i\}$ which sum to $\bar{U}$. Similar processes $\bar{V}$ and $\{\bar{V}^j\}$ may be built to bound the V^ε and $\{V^j\}$, respectively. We also can decompose $K = \sum_i K^{i,U} = \sum_j K^{j,V}$ according to the type of individual being killed. From hitting probabilities of Feller's branching diffusion $\bar{U}^i(1) = \langle \bar{U}^i, 1\rangle$, we know that with reasonably large probability one of the $\bar{U}^i$ clusters does hit 1, and we condition on such an event for a fixed choice of i, denoting the conditional law by Q_i. We now proceed in three steps:

Step 1. $K_{s_i+t}^{i,U}(1) \le t^{3/2-\varepsilon}$ for small t with reasonably large probability (see Lemma 4.3 below), uniformly in ε.

This step uses a modulus of continuity for the support of the dominating super-Brownian motions which states that they can spread locally no faster than $t^{1/2}$ with some logarithmic corrections which we omit for the purposes of this outline; see Theorem 3.5 in Mueller and Perkins (1992) for a more general version which we will need for the general γ case. This means both $\bar{U}^i$ and $\bar{V}^j$ are constrained to lie inside a growing space–time parabola rooted at their space–time birth points and hence the same is true for the dominated processes U^i and V^j. If τ_j is the lifetime of $\bar{V}^j$ then, using the known law of τ_j (it is the hitting time of zero by Feller's branching diffusion starting from ε) and a bit of geometry to see how large τ_j has to be for the parabola of $\bar{V}^j$ to intersect with that of $\bar{U}^i$ from s_i to $s_i + t$, one can easily deduce that with reasonably large probability the only $\bar{V}^j$ clusters which can intersect with the $\bar{U}^i$ cluster we have singled out are those born in the space–time rectangle $[s_i, s_i + t] \times [x_i - 2t^{1/2}, x_i + 2t^{1/2}]$. This means these are the only $K^{j,V}$'s [killing by descendants of (t_j, y_j)] that can contribute to $K^{i,U}$ on $[s_i, s_i + t]$ since other V particles will not collide with the U^i mass. In particular, with reasonably large probability none of the V^j clusters born before s_i can affect the mass of U^i on $[s_i, s_i + t]$; see Lemma 7.4 for the proof of this last assertion for general γ. The mean amount of killing by these V^j's can be no more than the mean amount of immigration which fuels these populations. More precisely if one integrates out the version of (1.10) for V^j over space, sums over the above indices j and bring the sum of the resulting K^j to the left-hand side, then one finds that if

$$R_i = [s_i, s_i + t] \times [x_i - 2t^{1/2}, x_i + 2t^{1/2}],$$

then

$$E\Bigg[\sum_{(t_j,y_j)\in R_i} K^j_{s_i+t}(1)\Bigg] \le E(\eta^-_\varepsilon([s_i, s_i+t] \times [x_i - 2t^{1/2}, x_i + 2t^{1/2}])) \le ct^{3/2}.$$

A standard interpolation argument now shows the integrand on the left-hand side is bounded by $ct^{3/2-\varepsilon}$ for small enough t a.s., and the claimed result follows from the above and the fact that any killing by $K^{i,U}$ is matched by a killing on V by one of the $K^{j,V}$'s. It will turn out that for $\gamma < 3/4$ one can get the same bound on $K^i_t(1)$.

Step 2. Under Q_i, which was the conditional law defined before step 1, $4\bar{U}^i_{s_i+\cdot}(1)$ is a 4-dimensional Bess^2-process and so $\bar{U}^i_{s_i+t}(1) \ge t^{1+\varepsilon}$ for small t a.s.

This follows from a standard change of measure argument; see Lemma 4.1 and its proof below. For general $\gamma < 3/4$, the mass $4\bar{U}^i_{s_i+\cdot}(1)$ will be a time change of a 4-dimensional Bess^2-process, and one will be able to show that $\bar{U}^i_{s_i+t}(1) \ge t^\beta$ for small t a.s. for some $\beta < 3/2$.

Step 3. There is a reasonably large Q_i-probability (uniform in ε) that $U^i_{s_i+t}(1) \ge t^{1+\varepsilon}$ for small t.

To see this, note that the above steps set up a competition between the conditioning which gives $\bar{U}^i(1)$ a positive linear drift and the killing which is limited by step 1. To decide which effect wins when considering $U^i(1)$, we will consider the ratio

$$R_t = \frac{\bar{U}^i_{s_i+t}(1) - U^i_{s_i+t}(1)}{\bar{U}^i_{s_i+t}(1)} \in [0,1]$$

of ghost particles to total population (alive and dead). An application of Itô's lemma will show that R is a submartingale satisfying

$$R_t = N_t + \frac{K_{s_i+t}(1)}{\bar{U}_{s_i+t}(1)},$$

where N_t is a continuous martingale. The last term is at most $t^{1/2-2\varepsilon}$ for small t with reasonably large Q_i probability by steps 1 and 2. We localize to get the above behavior almost surely up to a stopping time, take means and use Kolmogorov's inequality for martingales to see that R_t is less than $1/2$ with reasonably large probability, uniformly in ε. By step 2 we can conclude that on this set $U^i_{s_i+t}(1) \geq (1/2)t^{1+\varepsilon}$ for small t, and so $U^i_{s_i+t}(1)$ is bounded away from 0 for small t with reasonably large Q_i-probability uniformly in t, as required. This step is carried out in the proof of Proposition 3.2 in Section 5 below.

There are a number of problems when carrying out the above argument. In step 1 we should pay attention to the fact that the underlying probability is Q_i. In addition, the argument for general γ is more involved. For example, the clusters of the dominating processes $\bar{V}^j$ will no longer be independent as they are when $\gamma = 1/2$ due to the branching property of solutions. Also, the rate of propagation results in Mueller and Perkins (1992) only apply for solutions where there is an underlying historical process which records the ancestral histories of the surviving population members. We could extend the construction of our solutions to (1.9) and (1.10) to include such processes, but this gets a bit unwieldy. Instead we prove a comparison theorem for supports of solutions of parabolic SPDE's (Proposition 6.3) which allows us to derive these results from the corresponding property of solutions of (1.1) with $\sigma(u) = u^\gamma$. The latter property holds for any solution since these solutions are known to be unique in law by Mytnik (1998).

REMARK 1.3. The condition that $\gamma < 3/4$ is required in step 1 to ensure that with reasonably large probability, the V particles born before time s_i do not contribute to the killing. Such killing, if it occurred, could lead to the immediate annihilation of the ith seed with high probability. The bound on γ is also used in steps 2 and 3 since otherwise the lower bound on $\bar{U}^i_{s_i+t}(1)$ near 0 will be t^β for some $\beta > 3/2$ which will be of no use in keeping R_t small for t small.

Here is an outline of the paper. Section 2 gives a careful description of the approximating solutions arising in (1.9), (1.10) and the various decompositions of these processes. The actual construction of these approximate solutions is carried out in Appendix B, while the fact that limit points of these approximating solutions provide actual solutions to (1.6) is given in Appendix A, along with some standard moment bounds. In Section 3 an inclusion–exclusion argument reduces the nonuniqueness result to a pair of Propositions (3.2 and 3.3) which correspond to step 3 and an amalgamation of steps 1 and 2, respectively. In Section 4 Proposition 3.3 is then reduced to a sequence of 5 lemmas, the main ones being Lemma 4.1 and Lemma 4.3, corresponding to steps 2 and 1, respectively. Section 5 deals with the main parts of the proof rooted in stochastic analysis including the proofs of Lemma 4.1 and Proposition 3.2. Sections 6 and 7 deal with the main parts of the proof involving qualitative properties of the clusters including the proof of Lemma 4.3 (the growth rate of the killing measure) in Section 7. Section 7 also gives the proof of the comparison theorem for supports of solutions of certain SPDE's.

2. Set-up of equations. In what follows we assume that $\gamma \in [1/2, 3/4)$. We will carry out the method outlined in the Introduction.

Recall that $\mathbb{N}_\varepsilon = \{1, \dots, N_\varepsilon\}$ where $N_\varepsilon = \lfloor \varepsilon^{-1} \rfloor$. For any Polish space $\mathbf{E}$, let $D(\mathbb{R}_+, \mathbf{E})$ be the Skorokhod space of cadlag $\mathbf{E}$-valued paths with left limits in $\mathbf{E}$, and define

$$\begin{aligned} D^\varepsilon(\mathbb{R}_+, \mathbf{E}) &= D(\mathbb{R}_+, \mathbf{E}) \cap C(\mathbb{R}_+ \setminus \mathcal{G}_\varepsilon, \mathbf{E}) \\ &= \text{the space of cadlag } \mathbf{E}\text{-valued functions on } \mathbb{R}_+, \text{ whose paths} \\ &\qquad \text{are continuous on any time interval } \left[\frac{(i-1)\varepsilon}{2}, \frac{i\varepsilon}{2}\right), 1 \le i \le 2N_\varepsilon, \\ &\qquad \text{and on } [N_\varepsilon \varepsilon, \infty). \end{aligned}$$

We will construct a sequence of processes $\{(U^{i,\varepsilon}, V^{i,\varepsilon}), i \in \mathbb{N}_\varepsilon\}$ with sample paths in $(C(\mathbb{R}_+ \setminus \mathcal{G}_\varepsilon, C^+_{\mathrm{rap}}) \cap D^\varepsilon(\mathbb{R}_+, L^1(\mathbb{R}))^2$. For each $\phi \in C_b^2(\mathbb{R})$, w.p.1, U^i, V^j (we will suppress ε in our notation) will satisfy the following equations for all $t \ge 0$ and all $i, j \in \mathbb{N}_\varepsilon$. Recall that J^{x_i} was defined in (1.13):

$$(2.1)\quad \begin{cases} U_t^i(\phi) = \langle J^{x_i}, \phi\rangle \mathbf{1}(t \ge s_i) \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} U(s,x)^{\gamma - 1/2} U^i(s,x)^{1/2} \phi(x) W^{i,U}(ds, dx) \\ \qquad\qquad + \displaystyle\int_0^t U_s^i\left(\frac{1}{2}\Delta\phi\right) ds - K_t^{i,U}(\phi), \\ V_t^j(\phi) = \langle J^{y_j}, \phi\rangle \mathbf{1}(t \ge t_j) \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} V(s,x)^{\gamma - 1/2} V^j(s,x)^{1/2} \phi(x) W^{j,V}(ds, dx) \\ \qquad\qquad + \displaystyle\int_0^t V_s^j\left(\frac{1}{2}\Delta\phi\right) ds - K_t^{j,V}(\phi), \\ \qquad\qquad\qquad\qquad \text{with } U_t = \displaystyle\sum_i U_t^i, V_t = \sum_i V_t^i, \end{cases}$$

where, as will be shown in Proposition 2.1, U and V have paths in $D^\varepsilon(\mathbb{R}_+, C^+_{\mathrm{rap}})$. Here $W^{i,U}, W^{j,V}, i, j \in \mathbb{N}_\varepsilon$ are independent space time white noises. $K^{i,U}, K^{j,V}$ and hence K_t below, are all right-continuous nondecreasing $M_F(\mathbb{R})$-valued processes representing the mutual killing of the two kinds of particles, such that

$$(2.2)\qquad \sum_i K_t^{i,U} = \sum_j K_t^{j,V} =: K_t$$

and

$$(2.3)\qquad U_t(x) V_t(x) = 0 \qquad \forall t \ge 0, x \in \mathbb{R}.$$

That is, U and V have disjoint supports and hence the same is true of U^i and V^j for all $i,j\in\mathbb{N}_\varepsilon$. It follows from (2.1) with $\phi\equiv 1$ that for $t<s_i$, $K_t^{i,U}(1)+U_t^i(1)$ is a continuous nonnegative local martingale, hence supermartingale, starting at 0. Therefore $K_t^{i,U}=U_t^i=0, t<s_i$ and similarly $K_t^{j,V}=V_t^j=0, t<t_j$ for all $i,j\in\mathbb{N}_\varepsilon$. One can think of U and V as two populations with initial masses immigrating at times $s_i, i\in\mathbb{N}_\varepsilon$ and $t_j, j\in\mathbb{N}_\varepsilon$, respectively. Condition (2.3) implies the presence of a "hard killing" mechanism in which representatives of both populations annihilate each other whenever they meet. The meaning of the "hard killing" notion will become clearer when we will explain the construction of the equations as limits of so-called soft-killing models.

We can regard $K^{i,U}$ and $K^{j,V}$ as the "frozen" mass that was killed in corresponding populations due to the hard killing. If we reintroduce this mass back we should get the model without killing. To this end let us introduce the equations for "killed" populations which we denote by $\widetilde{U}^i,\widetilde{V}^j$. These will take values in the same path space as U^i, V^j. For each $\phi\in C_b^2(\mathbb{R})$, we require the following equations hold almost surely for all $t\ge 0$ and $i,j\in\mathbb{N}_\varepsilon$:

$$(2.4)\quad \begin{cases} \widetilde{U}_t^i(\phi)=\displaystyle\int_0^t\int_{\mathbb{R}}[(\widetilde{U}(s,x)+U(s,x))^{2\gamma}-U(s,x)^{2\gamma}]^{1/2} \\ \qquad\qquad \times\sqrt{\dfrac{\widetilde{U}^i(s,x)}{\widetilde{U}(s,x)}}\phi(x)\widetilde{W}^{i,U}(ds,dx) \\ \qquad\qquad +\displaystyle\int_0^t \widetilde{U}_s^i\biggl(\frac{1}{2}\Delta\phi\biggr)\,ds+K_t^{i,U}(\phi), \\ \widetilde{V}_t^j(\phi)=\displaystyle\int_0^t\int_{\mathbb{R}}[(\widetilde{V}(s,x)+V(s,x))^{2\gamma}-V(s,x)^{2\gamma}]^{1/2} \\ \qquad\qquad \times\sqrt{\dfrac{\widetilde{V}^j(s,x)}{\widetilde{V}(s,x)}}\phi(x)\widetilde{W}^{j,V}(ds,dx) \\ \qquad\qquad +\displaystyle\int_0^t \widetilde{V}_s^j\biggl(\frac{1}{2}\Delta\phi\biggr)\,ds+K_t^{j,V}(\phi), \\ \qquad\qquad\qquad\qquad \text{with } \widetilde{U}_t=\displaystyle\sum_i\widetilde{U}_t^i, \widetilde{V}_t=\sum_j\widetilde{V}_t^j, \end{cases}$$

where, as will be shown in Proposition 2.1, $\widetilde{U}$ and $\widetilde{V}$ have paths in $D^\varepsilon(\mathbb{R}_+, C_{\mathrm{rap}}^+)$ and we define $\sqrt{0/0}=0$ in the stochastic integral. The white noises $\widetilde{W}^{i,U}$, $\widetilde{W}^{j,V}$, $i,j\in\mathbb{N}_\varepsilon$, are independent and also independent of $\{W^{i,U},W^{j,V}, i,j\in\mathbb{N}_\varepsilon\}$. Again it is easy to see that

$$(2.5)\qquad \widetilde{U}_t^i=0 \qquad \text{for } t<s_i \quad\text{and}\quad \widetilde{V}_t^j=0 \qquad \text{for } t<t_j, i,j\in\mathbb{N}_\varepsilon.$$

Then using stochastic calculus, we deduce that the processes defined by $\bar{U}_t^i\equiv U_t^i+\widetilde{U}_t^i, \bar{V}^i\equiv V_t^i+\widetilde{V}_t^i$ satisfy the following equations for each ϕ as above, w.p.1 for all $t \ge 0$, $i, j \in \mathbb{N}_\varepsilon$:

$$
(2.6)\quad \begin{cases}
\bar U_t^i(\phi) = \langle J^{x_i}, \phi\rangle \mathbf{1}(t \ge s_i) + \int_0^t \bar U_s^i\left(\frac{1}{2}\Delta\phi\right) ds \\
\qquad + \int_0^t \int_{\mathbb{R}} \sqrt{U(s,x)^{2\gamma-1}U^i(s,x) + (\bar U(s,x)^{2\gamma} - U(s,x)^{2\gamma})\frac{\widetilde U^i(s,x)}{\widetilde U(s,x)}} \\
\qquad\qquad \times \phi(x)\bar W^{i,U}(ds,dx), \\
\bar V_t^j(\phi) = \langle J^{y_j}, \phi\rangle \mathbf{1}(t \ge t_j) + \int_0^t \bar V_s^j\left(\frac{1}{2}\Delta\phi\right) ds \\
\qquad + \int_0^t \int_{\mathbb{R}} \sqrt{V(s,x)^{2\gamma-1}V^j(s,x) + (\bar V(s,x)^{2\gamma} - V(s,x)^{2\gamma})\frac{\widetilde V^j(s,x)}{\widetilde V(s,x)}} \\
\qquad\qquad \times \phi(x)\bar W^{j,V}(ds,dx), \\
\qquad\qquad\qquad \text{with } \bar U_t = \sum_i \bar U_t^i, \bar V_t = \sum_j \bar V_t^j,
\end{cases}
$$

where, $\{\bar W^{i,U}, \bar W^{j,V}, i, j \in \mathbb{N}_\varepsilon\}$ is again a collection of independent white noises. In spite of the complicated appearance of (2.6), for $\bar U, \bar V$ we easily get

$$
(2.7)\quad \begin{cases}
\bar U_t(\phi) = \int_0^t \int \phi(x)\eta_\varepsilon^+(ds,dx) + \int_0^t \bar U_s\left(\frac{1}{2}\Delta\phi\right) ds \\
\qquad + \int_0^t \int_{\mathbb{R}} \bar U(s,x)^\gamma \phi(x)\bar W^U(ds,dx), \qquad t \ge 0, \\
\bar V_t(\phi) = \int_0^t \int \phi(x)\eta_\varepsilon^-(ds,dx) + \int_0^t \bar V_s\left(\frac{1}{2}\Delta\phi\right) ds \\
\qquad + \int_0^t \int_{\mathbb{R}} \bar V(s,x)^\gamma \phi(x)\bar W^V(ds,dx), \qquad t \ge 0,
\end{cases}
$$

for independent white noises $\bar W^U$ and $\bar W^V$. One can easily derive from the proof of Theorem 1 of Mytnik (1998) that $(\bar U, \bar V)$ is unique in law (see Remark A.2 below).

Our next proposition establishes existence of solutions to the above systems of equations. The filtration $(\mathcal{F}_t)$ will always be right-continuous and such that $\mathcal{F}_0$ contains the P-null sets in $\mathcal{F}$. For any $T \ge 1$, the space $D^\varepsilon([0,T], \mathbf{E})$ is defined in the same way as $D^\varepsilon(\mathbb{R}_+, \mathbf{E})$, but for $\mathbf{E}$-valued functions on $[0,T]$.

For any function $f \in D(\mathbb{R}_+, \mathbb{R})$, we set $\mathbf{\Delta} f(t) \equiv f(t) - f(t-)$, for any $t \ge 0$.

Proposition 2.1. *There exists a sequence* $(U^i, V^i, \widetilde U^i, \widetilde V^i, \bar U^i, \bar V^i, K^{i,U}, K^{i,V})_{i\in\mathbb{N}_\varepsilon}$ *of processes in*

$$
((C([0,T] \setminus \mathcal{G}_\varepsilon, C_{\text{rap}}^+) \cap D^\varepsilon([0,T], L^1(\mathbb{R})))^4 \times D^\varepsilon(\mathbb{R}_+, C_{\text{rap}}^+)^2 \times D^\varepsilon(\mathbb{R}_+, M_F(\mathbb{R}))^2)^{N_\varepsilon},
$$

which satisfies (2.1)–(2.7). Moreover, $(U, V, \widetilde{U}, \widetilde{V}) \in D^\varepsilon(\mathbb{R}_+, C^+_{\mathrm{rap}})^4$, *and the following conditions hold:*

(a) *For any* $i \in \mathbb{N}_\varepsilon$, $\bar{U}^i_{s_i+\cdot} \in C(\mathbb{R}_+, C^+_{\mathrm{rap}})$, $\bar{V}^i_{t_i+\cdot} \in C(\mathbb{R}_+, C^+_{\mathrm{rap}})$ *and*

$$\bar{U}^i(s, \cdot) = 0, \qquad s < s_i, \qquad \bar{V}^i(s, \cdot) = 0, \qquad s < t_i.$$

(b) $K.$ *only has jumps at times in* $\mathcal{G}_\varepsilon$, *and*

$$\sup_t \mathbf{\Delta} K_t(1) \le \varepsilon. \tag{2.8}$$

In what follows we will call $\bar{U}^i$, $\bar{V}^i$ (resp., U^i, V^i) the clusters of the processes $\bar{U}$, $\bar{V}$ (resp., U, V).

Now with all the processes in hand, let us state the results which will imply the nonuniqueness in (1.6) with zero initial conditions. First define

$$u_\varepsilon(t) := U_t - V_t \in C_{\mathrm{rap}} \tag{2.9}$$

and recall that U_t, V_t implicitly depend on ε. Then it is easy to see from the above construction that u_ε satisfies the following SPDE:

$$\begin{aligned} \langle u_\varepsilon(t), \phi \rangle &= \sum_i \langle J^{x_i} \mathbf{1}(t \ge s_i), \phi \rangle - \sum_j \langle J^{y_j} \mathbf{1}(t \ge t_j), \phi \rangle \\ &\quad + \int_0^t \frac{1}{2} \langle u_\varepsilon(s), \Delta\phi \rangle \, ds + \int_0^t \int |u_\varepsilon(s,x)|^\gamma \phi(x) W(ds, dx) \end{aligned} \tag{2.10}$$

for $\phi \in C_b^2(\mathbb{R})$.

The following two propositions will imply Theorem 1.1.

PROPOSITION 2.2. *Let* $\varepsilon_n = \frac{1}{n}$. *Then* $\{u_{\varepsilon_n}\}_n$ *is tight in* $D(\mathbb{R}_+, C_{\mathrm{rap}})$. *If* u *is any limit point as* $\varepsilon_{n_k} \downarrow 0$, *then* u *is a* C_{rap}*-valued solution of the SPDE (1.6).*

The next proposition is just a restatement of Claim 1.2.

PROPOSITION 2.3. *There exists* $\delta_{2.3}, \varepsilon_{2.3} > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_{2.3}]$,

$$P\left(\sup_{t \in [0,1]} \int U_t^\varepsilon(x)\, dx > \delta_{2.3} \right) > \delta_{2.3}.$$

The proof of Proposition 2.2 will be standard and may be found in Appendix A. Most of the paper is devoted to the proof of Proposition 2.3.

3. Outline of the proof of Proposition 2.3. We analyze the behavior of the clusters U^i, V^i and show that with positive probability at least one of them survives. As in the previous section, we suppress dependence on the parameter $\varepsilon \in (0, 1]$.

To make our analysis precise we need to introduce the event A_i that the mass of the cluster $\bar{U}^i$ reaches 1 before the cluster dies. Define

$$\bar{\tau}_i = \inf\{t : \bar{U}^i_{s_i+t}(1) = 1\},$$
$$A_i \equiv \{\bar{\tau}_i < \infty\},$$

so that $\bar{\tau}_i$ is an $(\mathcal{F}_{s_i+t})$-stopping time. Since we will often assume that one of A_i occurs with positive probability, we define the conditional probability measure Q_i,

$$Q_i(A) = P(A|A_i) \qquad \forall A \in \mathcal{F}. \tag{3.1}$$

We need the following elementary lemma whose proof is given in Section 5.

LEMMA 3.1. *For all $1 \le i, j \le N_\varepsilon$, the events $A_i = A_i(\varepsilon)$ satisfy:*

(a) $P(A_i) = \varepsilon$;
(b) $P(A_i \cap A_j) = \varepsilon^2, i \neq j$.

A simple inclusion–exclusion lower bound on $P(\bigcup_{i=1}^{\lfloor 2^{-1}\varepsilon^{-1}\rfloor} A_i)$ shows that for $\varepsilon \le 1/4$, with probability at least 3/16, at least one cluster of $\bar{U}^i$ survives until it attains mass 1. We will focus on the corresponding U^i and to show it is nonzero with positive probability (all uniformly in ε), and we will establish a uniform (in ε) escape rate. Set

$$\beta = \frac{3/2 - \gamma}{2(1-\gamma)}, \tag{3.2}$$

and note that $\beta < 3/2$ for $\gamma < 3/4$. Our escape rate depends on a parameter $\delta_1 \in (0,1)$ (which will eventually be taken small enough depending on γ) and is given in the event

$$B_i(t) = \{U^i_{s_i+s}(1) \ge \tfrac{1}{2} s^{\beta+\delta_1}, \forall s \in [\varepsilon^{2/3}, t]\}.$$

Denote the closed support of a measure μ on $\mathbb{R}$ by $S(\mu)$. Let

$$T_R = \inf\{t : \|\bar{U}_t(\cdot)\|_\infty \vee \|\bar{V}_t(\cdot)\|_\infty > R\},$$

so that $(T_R - s_i)^+$ is an $\mathcal{F}_{s_i+t}$-stopping time. To localize the above escape rate we let $\delta_0 \in (0, 1/4]$ and define additional $(\mathcal{F}_{s_i+t})$-stopping times ($\inf \varnothing = \infty$) by

$$\begin{aligned}
\rho_i^{\delta_0,\varepsilon} &= \rho_i = \inf\{t : S(\bar{U}^i_{s_i+t}) \not\subset [x_i - \varepsilon^{1/2} - t^{1/2-\delta_0}, x_i + \varepsilon^{1/2} + t^{1/2-\delta_0}]\},\\
H_i^{\delta_1,\varepsilon} &= H_i = \inf\{t \ge 0 : \bar{U}^i_{t+s_i}(1) < (t+\varepsilon)^{\beta+\delta_1}\},\\
\theta_i^{\delta_0,\varepsilon} &= \theta_i = \inf\{t : K^{i,U}_{t+s_i}(1) > (t+\varepsilon)^{3/2-2\delta_0}\},\\
v_i^{\delta_0,\delta_1,\varepsilon} &= v_i = \bar{\tau}_i \wedge H_i \wedge \theta_i \wedge \rho_i \wedge (T_R - s_i)^+.
\end{aligned}$$

We now state the two key results and show how they lead to Proposition 2.3. The first result is proved in Section 5 below using some stochastic analysis and change of measure arguments. The second is reduced to a sequence of lemmas in Section 4.

PROPOSITION 3.2. *There are $\delta_{3.2}(\gamma) > 0$ and $p = p_{3.2}(\gamma) \in (0, 1/2]$ such that if $0 < 2\delta_0 \le \delta_1 \le \delta_{3.2}$, then*

$$Q_i(B_i(t \wedge v_i)) \ge 1 - 5t^p \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1].$$

PROPOSITION 3.3. *For each $\delta_1 \in (0, 1)$ and small enough $\delta_0 > 0$, depending on δ_1 and γ, there exists a nondecreasing function $\delta_{3.3}(t)$, not depending on ε, such that*

$$\lim_{t \downarrow 0} \delta_{3.3}(t) = 0,$$

and for all $\varepsilon, t \in (0, 1]$,

$$P\left(\bigcup_{i \ge 1}^{tN_\varepsilon} (\{v_i < t\} \cap A_i)\right) \le t\delta_{3.3}(t).$$

With these two propositions we can give the following:

PROOF OF PROPOSITION 2.3. Let $p = p_{3.2}$ and $\delta(t) = \delta_{3.3}(t)$. Assume $t = t_{2.3} \in (0, 1]$ is chosen so that $5t^p + t + \delta(t) \le 1/2$. We claim that

$$P\left(\bigcup_{i=1}^{tN_\varepsilon} B_i(t)\right) \ge \frac{t}{4} \qquad \forall \varepsilon \in (0, t/8]. \tag{3.3}$$

Choose $\delta_1 > 0$ as in Proposition 3.2, then $\delta_0 \in (0, \delta_1/2]$ as in Proposition 3.3 and finally $t = t_{2.3}$ as above. Then we have

$$\begin{aligned}
P\left(\bigcup_{i=1}^{tN_\varepsilon} B_i(t)\right) &\ge P\left(\bigcup_{i=1}^{tN_\varepsilon} B_i(t \wedge v_i) \cap A_i \cap \{v_i \ge t\}\right) \\
&\ge P\left(\bigcup_{i=1}^{tN_\varepsilon} B_i(t \wedge v_i) \cap A_i\right) - P\left(\bigcup_{i=1}^{tN_\varepsilon} A_i \cap \{v_i < t\}\right) \\
&\ge \sum_{i=1}^{tN_\varepsilon} P(B_i(t \wedge v_i) \cap A_i) - \sum_{i=1}^{tN_\varepsilon} \sum_{j=1, j \ne i}^{tN_\varepsilon} P(A_i \cap A_j) \\
&\quad - P\left(\bigcup_{i=1}^{tN_\varepsilon} A_i \cap \{v_i < t\}\right).
\end{aligned}$$

Recall the definition of the conditional law Q_i, and use Lemma 3.1(b) to see that the above is at least

$$\sum_{i=1}^{tN_\varepsilon} Q_i(B_i(t\wedge v_i))P(A_i) - t^2N_\varepsilon^2\varepsilon^2 - P\left(\bigcup_{i=1}^{tN_\varepsilon} A_i\cap\{v_i<t\}\right)$$
$$\geq \varepsilon(tN_\varepsilon-1) - 5N_\varepsilon\varepsilon t^{1+p_{3.2}} - t^2 - t\delta_{3.3}(t)$$
$$\geq t[1-5t^{p_{3.2}} - t - \delta_{3.3}(t)] - 2\varepsilon,$$

where the next to last inequality follows by Lemma 3.1(a) and Propositions 3.2, 3.3. Our choice of $t=t_{2.3}$ shows that for $\varepsilon\leq t_{2.3}/8$. The above is at least $\frac{t}{2}-\frac{t}{4}=\frac{t}{4}$. It follows from the final part of (2.1) that for all $t\geq 0$, $\int U_t^\varepsilon(x)\,dx \geq \max_i \int U_t^{i,\varepsilon}(x)\,dx$. The proposition follows immediately from (3.3). $\square$

4. Lower bounds on the stopping times: Proof of Proposition 3.3. In this section we reduce the proof of Proposition 3.3 to five lemmas which will be proved in Sections 5–7 below. The bounds in this section may depend on the parameters δ_0 and δ_1, but not ε. We introduce

$$\bar\delta = \bar\delta(\gamma) = \tfrac{1}{3}(\tfrac{3}{2}-2\gamma)\in(0,1/6]. \tag{4.1}$$

Lemma 4.1. *For $\delta_0>0$ sufficiently small, depending on δ_1,γ, there is a function $\eta_{4.1}:\mathbb{R}_+\to[0,1]$ so that $\eta_{4.1}(t)\to 0$ as $t\downarrow 0$, and for all $t>0$ and $\varepsilon\in(0,1]$,*

$$Q_i(H_i\leq \bar\tau_i\wedge\rho_i\wedge t)\leq \eta_{4.1}(t) + 8\varepsilon^{\delta_1}.$$

Lemma 4.2. *For all $t>0$ and $\varepsilon\in(0,1]$,*

$$Q_i(\bar\tau_i\leq t\wedge(T_R-s_i)^+)\leq 2\gamma R^{2\gamma-1}t+\varepsilon.$$

Lemma 4.3. *If $0<\delta_0\leq\bar\delta$, there is a constant $c_{4.3}$, depending on γ and δ_0, so that*

$$Q_i(\theta_i<\rho_i\wedge t)\leq c_{4.3}(t\vee\varepsilon)^{\delta_0}\qquad \textit{for all } \varepsilon,t\in(0,1] \textit{ and } s_i\leq t.$$

It remains to handle the ρ_i and T_R. This we do under the probability P.

Lemma 4.4. *There is a constant $c_{4.4}\geq 1$, depending on γ and δ_0, so that*

$$P\left(\bigcup_{i=1}^{pN_\varepsilon}\{\rho_i\leq t\}\right)\leq c_{4.4}(t\vee\varepsilon)p\mathbf{1}(p\geq\varepsilon)\quad \textit{for all } \varepsilon,p,t\in(0,1].$$

LEMMA 4.5. *For any $\varepsilon_0 > 0$ there is a function $\delta_{4.5} : (0,2] \to \mathbb{R}_+$ so that $\lim_{t\to 0} \delta_{4.5}(t) = 0$ and*

$$P\Big(\sup_{s<t, x\in\mathbb{R}} \bar{U}(s,x) \vee \bar{V}(s,x) > t^{-2-\varepsilon_0}\Big) \le t\delta_{4.5}(t) \qquad \textit{for all } \varepsilon \in (0,1], t \in (0,2].$$

Assuming the above five results it is now very easy to give the following:

PROOF OF PROPOSITION 3.3. For $\delta_1 \in (0,1)$ choose $\delta_0 > 0$ small enough so that the conclusion of Lemmas 4.1 and 4.3 hold. Then for $0 < t \le 1 \le R$ and $0 < \varepsilon \le 1$, using Lemma 4.4 with $p = t$, we have

$$\begin{aligned}
&P\left(\bigcup_{i=1}^{tN_\varepsilon} \{v_i < t\} \cap A_i\right) \\
&\quad \le P\left(\bigcup_{i=1}^{tN_\varepsilon} \{T_R < t + s_i\}\right) + P\left(\bigcup_{i=1}^{tN_\varepsilon} \{\bar{\tau}_i < t \wedge (T_R - s_i)^+\} \cap A_i\right) \\
&\qquad + P\left(\bigcup_{i=1}^{tN_\varepsilon} \{\rho_i < t\}\right) + P\left(\bigcup_{i=1}^{tN_\varepsilon} \{H_i < \bar{\tau}_i \wedge \rho_i \wedge t\} \cap A_i\right) \\
&\qquad + P\left(\bigcup_{i=1}^{tN_\varepsilon} \{\theta_i < \rho_i \wedge t\} \cap A_i\right) \\
&\quad \le P(T_R < 2t) + \sum_{i=1}^{tN_\varepsilon} Q_i(\bar{\tau}_i \le t \wedge (T_R - s_i)^+) P(A_i) + c_{4.4}(t \vee \varepsilon) t \mathbf{1}(t \ge \varepsilon) \\
&\qquad + \sum_{i=1}^{tN_\varepsilon} Q_i(H_i < \bar{\tau}_i \wedge \rho_i \wedge t) P(A_i) + \sum_{i=1}^{tN_\varepsilon} Q_i(\theta_i < \rho_i \wedge t) P(A_i).
\end{aligned}$$

Now apply Lemma 3.1 and Lemmas 4.1–4.3 to bound the above by

$$\begin{aligned}
&P\Big(\sup_{s<2t, x\in\mathbb{R}} \bar{U}(s,x) \vee \bar{V}(s,x) > R\Big) + 2\gamma R^{2\gamma-1} t^2 + \varepsilon t + c_{4.4} t^2 \\
&\qquad + t\eta_{4.1}(t) + t8\varepsilon^{\delta_1} + tc_{4.3}(t \vee \varepsilon)^{\delta_0}.
\end{aligned}$$

We may assume without loss of generality that $\eta_{4.1}$ is nondecreasing and $t \ge \varepsilon$ (or else the left-hand side is 0). Set $R = t^{-2-\varepsilon_0}$, where $\varepsilon_0 > 0$ is chosen so that $3 - 4\gamma - \varepsilon_0(2\gamma - 1) > 0$ and use Lemma 4.5 to obtain the required bound with

$$\delta_{3.3}(t) = 2\delta_{4.5}(2t) + 2\gamma(2t)^{3-4\gamma-\varepsilon_0(2\gamma-1)} + 2c_{4.4}t + \eta_{4.1}(t) + 8t^{\delta_1} + c_{4.3}t^{\delta_0}.$$

This finishes the proof of Proposition 3.3. □

5. Change of measure and stochastic analysis: Proofs of Proposition 3.2 and Lemmas 4.1 and 4.2. Define

$$\bar{\tau}_i(0) = \inf\{t \geq 0 : \bar{U}^i_{s_i+t}(1) = 0\}$$

and

$$\bar{\tau}_i(0,1) = \bar{\tau}_i(0) \wedge \bar{\tau}_i,$$

where $\bar{\tau}_i$ was defined at the beginning of Section 3.

It follows from (2.6) that

$$\bar{U}^i_{t+s_i}(1) = \varepsilon + \bar{M}^i_t, \tag{5.1}$$

where $\bar{M}^i$ is a continuous local $(\mathcal{F}_{s_i+t})$-martingale starting at 0 at $t = 0$ and satisfying

$$\begin{aligned}\langle \bar{M}^i\rangle_t = \int_{s_i}^{s_i+t} \int & U(s,x)^{2\gamma-1} U^i(s,x) \\ & + (\bar{U}(s,x)^{2\gamma} - U(s,x)^{2\gamma}) \frac{\widetilde{U}^i(s,x)}{\widetilde{U}(s,x)}\,dx\,ds.\end{aligned} \tag{5.2}$$

LEMMA 5.1. *There is a* $c_{5.1} = c_{5.1}(\gamma) > 0$ *so that*

$$P(\bar{\tau}_i(0) > t) \leq c_{5.1}\varepsilon^{2-2\gamma} t^{-1} \qquad \textit{for all } t > 0.$$

PROOF. It follows from (5.2) that

$$\begin{aligned}
&\frac{d\langle \bar{M}^i\rangle(t)}{dt} \\
&\quad = \int U(s_i+t,x)^{2\gamma-1} U^i(s_i+t,x) \\
&\qquad\quad + (\bar{U}(s_i+t,x)^{2\gamma} - U(s_i+t,x)^{2\gamma}) \frac{\widetilde{U}^i(s_i+t,x)}{\widetilde{U}(s_i+t,x)}\,dx \\
&\quad \geq \int U(s_i+t,x)^{2\gamma-1} U^i(s_i+t,x) + \widetilde{U}(s_i+t,x)^{2\gamma-1}\widetilde{U}^i(s_i+t,x)\,dx \\
&\quad \geq \int U^i(s_i+t,x)^{2\gamma} + \widetilde{U}^i(s_i+t,x)^{2\gamma}\,dx \\
&\quad \geq 2^{1-2\gamma} \int \bar{U}^i(s_i+t,x)^{2\gamma}\,dx.
\end{aligned} \tag{5.3}$$

If $\gamma > 1/2$, the result now follows from Lemma 3.4 of Mueller and Perkins (1992).

If $\gamma = 1/2$, then one can construct a time scale τ_t satisfying $\tau_t \le t$ for $\tau_t \le \bar{\tau}_i(0)$, under which $t \to U^i_{s_i+\tau_t}(1)$ becomes Feller's continuous state branching diffusion. The required result then follows from well-known bounds on the extinction time for the continuous state branching process; for example, see equation (II.5.12) in Perkins (2002). □

PROPOSITION 5.2.

$$Q_i(A) = \int_A \frac{\bar{U}^i_{s_i+(\bar{\tau}_i\wedge t)}(1)}{\varepsilon}\,dP \qquad \textit{for all } A \in \mathcal{F}_{s_i+t}, t \ge 0.$$

PROOF. Since $\bar{\tau}_i(0,1) < \infty$ a.s. (by the previous lemma) and $\bar{U}^i(1)$ remains at 0 when it hits 0, we have

$$\mathbf{1}(\bar{\tau}_i < \infty) = \bar{U}^i_{s_i+\bar{\tau}_i(0,1)}(1) \qquad \text{a.s.} \tag{5.4}$$

By considering $\bar{\tau}_i(0,1) \le t$ and $\bar{\tau}_i(0,1) > t$ separately we see that

$$\bar{U}^i_{s_i+(\bar{\tau}_i(0,1)\wedge t)}(1) = \bar{U}^i_{s_i+(\bar{\tau}_i\wedge t)}(1) \qquad \text{a.s. on } \{\bar{\tau}_i > t\}. \tag{5.5}$$

If $A \in \mathcal{F}_{s_i+t}$, then

$$\begin{aligned} P(A, \bar{\tau}_i < \infty) &= P(A, \bar{\tau}_i \le t) + P(A, t < \bar{\tau}_i < \infty) \\ &= \int \mathbf{1}(A, \bar{\tau}_i \le t)\bar{U}^i_{s_i+(\bar{\tau}_i\wedge t)}(1)\,dP \\ &\quad + E(\mathbf{1}(A, \bar{\tau}_i > t)P(\bar{\tau}_i < \infty|\mathcal{F}_{s_i+t})). \end{aligned} \tag{5.6}$$

By (5.4) and (5.5) on $\{\bar{\tau}_i > t\}$,

$$\begin{aligned} P(\bar{\tau}_i < \infty|\mathcal{F}_{s_i+t}) &= E(\bar{U}^i_{s_i+\bar{\tau}_i(0,1)}(1)|\mathcal{F}_{s_i+t}) \\ &= \bar{U}^i_{s_i+(\bar{\tau}_i(0,1)\wedge t)}(1) \\ &= \bar{U}^i_{s_i+(\bar{\tau}_i\wedge t)}(1). \end{aligned}$$

Then from (5.6) we conclude that

$$P(A, \bar{\tau}_i < \infty) = \int_A \bar{U}^i_{s_i+(\bar{\tau}_i\wedge t)}(1)\,dP. \tag{5.7}$$

If $A = \Omega$, we get

$$P(\bar{\tau}_i < \infty) = E(\bar{U}^i_{s_i+(\bar{\tau}_i\wedge t)}(1)) = \bar{U}^i_{s_i}(1) = \varepsilon. \tag{5.8}$$

Taking ratios in the last two equalities, we see that

$$Q_i(A) = \int_A \bar{U}^i_{s_i+(\bar{\tau}_i\wedge t)}(1)/\varepsilon\,dP$$

as required. □

PROOF OF LEMMA 3.1. (a) Immediate from (5.8).

(b) Assume $i < j$. The orthogonality of the bounded continuous $(\mathcal{F}_t)$-martingales $\bar{U}^i_{t\wedge(s_i+\bar{\tau}^i(0,1))}(1)$ and $\bar{U}^j_{t\wedge(t_j+\bar{\tau}^j(0,1))}(1)$ [see (2.6)] shows that

$$\begin{aligned} &E[\bar{U}^i_{s_i+\bar{\tau}^i(0,1)}(1)\bar{U}^j_{s_j+\bar{\tau}^j(0,1)}(1)|\mathcal{F}_{s_j}]\mathbf{1}(s_i+\bar{\tau}_i(0,1)>s_j) \\ &\qquad = \bar{U}^i_{s_j}(1)\varepsilon\mathbf{1}(s_i+\bar{\tau}_i(0,1)>s_j). \end{aligned} \tag{5.9}$$

By first using (5.4) and then (5.9), we have

$$\begin{aligned} &P(A_i\cap A_j) \\ &\quad = E[\bar{U}^i_{s_i+\bar{\tau}^i(0,1)}(1)\bar{U}^j_{s_j+\bar{\tau}^j(0,1)}(1)] \\ &\quad = E[\bar{U}^i_{s_i+\bar{\tau}^i(0,1)}(1)\mathbf{1}(s_i+\bar{\tau}_i(0,1)\le s_j)E[\bar{U}^j_{s_j+\bar{\tau}^j(0,1)}(1)|\mathcal{F}_{s_j}]] \\ &\qquad + E[E[\bar{U}^i_{s_i+\bar{\tau}^i(0,1)}(1)\bar{U}^j_{s_j+\bar{\tau}^j(0,1)}(1)|\mathcal{F}_{s_j}]\mathbf{1}(s_i+\bar{\tau}_i(0,1)>s_j)] \\ &\quad = E[\bar{U}^i_{(s_i+\bar{\tau}^i(0,1))\wedge s_j}(1)\mathbf{1}(s_i+\bar{\tau}_i(0,1)\le s_j)\varepsilon] \\ &\qquad + E[\bar{U}^i_{s_j}(1)\varepsilon\mathbf{1}(s_i+\bar{\tau}_i(0,1)>s_j)] \\ &\quad = E[\bar{U}^i_{(s_i+\bar{\tau}^i(0,1))\wedge s_j}(1)]\varepsilon = \varepsilon^2. \end{aligned}$$

□

PROOF OF LEMMA 4.1. Clearly $\bar{M}^i_{t\wedge\bar{\tau}_i}$ is a bounded $(\mathcal{F}_{s_i+t})$-martingale under P. Girsanov's theorem [see Theorem VIII.1.4 of Revuz and Yor (1999)] shows that

$$\bar{M}^i_{t\wedge\bar{\tau}_i} = \bar{M}^{i,Q}_t + \int_0^{t\wedge\bar{\tau}_i} \bar{U}^i_{s_i+s}(1)^{-1}\,d\langle\bar{M}^i\rangle_s, \tag{5.10}$$

where $\bar{M}^{i,Q}$ is an $(\mathcal{F}_{s_i+t})$-local martingale under Q_i such that $\langle\bar{M}^{i,Q}\rangle_t = \langle\bar{M}^i\rangle_{t\wedge\bar{\tau}_i}$.

If $\bar{X}_t = \bar{U}^i_{s_i+(t\wedge\bar{\tau}_i)}(1)$, for

$$t \le \int_0^{\bar{\tau}_i} \bar{X}_s^{-1}\,d\langle\bar{M}^i\rangle_s \equiv R_i,$$

define τ_t by

$$\int_0^{\tau_t} \bar{X}_s^{-1}\,d\langle\bar{M}^i\rangle_s = t. \tag{5.11}$$

Since $\bar{X}_s > 0$ and $\frac{d\langle\bar{M}^i\rangle_s}{ds} > 0$ for all $0 \le s \le \bar{\tau}_i$ Q_i-a.s. [see (5.2)] this uniquely defines τ under Q_i as a strictly increasing continuous function on $[0, R_i] = [0, \tau^{-1}(\bar{\tau}_i)]$. By differentiating (5.11) we see that

$$\frac{d}{dt}(\langle\bar{M}^i\rangle\circ\tau)(t) = \bar{X}(\tau_t), \qquad t \le \tau^{-1}(\bar{\tau}_i). \tag{5.12}$$

Let $N_t = \bar{M}^{i,Q}(\tau_t)$, so that

$$Z_t \equiv \bar{X}(\tau_t) = \varepsilon + N_t + t \qquad \text{for } t \le \tau^{-1}(\bar{\tau}_i),$$

and by (5.12) for t as above,

$$\langle N \rangle_t = \langle \bar{M}^i \rangle(\tau_t) = \int_0^t Z_s \, ds.$$

Therefore we can extend the continuous local martingale $N(t \wedge \tau^{-1}(\bar{\tau}_i))$ for $t > \tau^{-1}(\bar{\tau}_i)$ so that $4Z_t$ is the square of a 4-dimensional Bessel process; see Section XI.1 of Revuz and Yor (1999). By the escape rate for $4Z$ [see Theorem 5.4.6 of Knight (1981)] and a comparison theorem for SDE [Theorem V.43.1 of Rogers and Williams (1987)] there is a nondecreasing $\eta_{\delta_0} : \mathbb{R}_+ \to [0,1]$ so that $\eta_{\delta_0}(0+) = 0$ and if $T_Z = \inf\{t : Z_t = 1\}$, and

$$\Gamma(\varepsilon, \delta_0) = \inf_{0 < t \le T_Z} \frac{Z(t)}{t^{1+\delta_0}},$$

then

$$\sup_{0 < \varepsilon \le 1} Q_i(\Gamma(\varepsilon, \delta_0) \le r) \le \eta_{\delta_0}(r). \tag{5.13}$$

Clearly $T_Z = \tau^{-1}(\bar{\tau}_i)$ and so

$$\inf_{0 < u \le \bar{\tau}_i} \frac{\bar{X}(u)}{\tau^{-1}(u)^{1+\delta_0}} = \inf_{0 < t \le T_Z} \frac{\bar{X}(\tau_t)}{t^{1+\delta_0}} = \Gamma(\varepsilon, \delta_0).$$

That is,

$$\bar{X}(u) \ge \Gamma(\varepsilon, \delta_0) \tau^{-1}(u)^{1+\delta_0} \qquad \text{for all } 0 < u \le \bar{\tau}_i. \tag{5.14}$$

To get a lower bound on $\tau^{-1}(u)$, use (5.3) to see that for $s < \rho_i \wedge \bar{\tau}_i$,

$$\begin{aligned}
\frac{d\langle \bar{M}^i \rangle_s}{ds} &\ge 2^{1-2\gamma} \int \mathbf{1}(x_i - \varepsilon^{1/2} - s^{(1/2)-\delta_0} \le x \le x_i + \varepsilon^{1/2} + s^{(1/2)-\delta_0}) \\
&\qquad \times \bar{U}^i(s_i + s, x)^{2\gamma} \, dx \\
&\ge 2^{1-2\gamma} [2(\varepsilon^{1/2} + s^{(1/2)-\delta_0})]^{1-2\gamma} \bar{X}(s)^{2\gamma},
\end{aligned}$$

where the bound on s is used in the last line. Therefore for $\varepsilon/2 \le s < \rho_i \wedge \bar{\tau}_i$ there is a $c_1(\gamma) > 0$ so that

$$\begin{aligned}
\frac{d\tau^{-1}(s)}{ds} &= \bar{X}_s^{-1} \frac{d\langle \bar{M}^i \rangle_s}{ds} \\
&\ge c_1(\gamma) s^{((1/2)-\delta_0)(1-2\gamma)} \bar{X}_s^{2\gamma-1} \\
&\ge c_1(\gamma) \Gamma(\varepsilon, \delta_0)^{2\gamma-1} s^{((1/2)-\delta_0)(1-2\gamma)} \tau^{-1}(s)^{(2\gamma-1)(1+\delta_0)},
\end{aligned}$$

where (5.14) is used in the last line. Therefore if $\varepsilon \le t \le \rho_i \wedge \bar{\tau}_i$, then

$$\int_{\varepsilon/2}^{t} \frac{d\tau^{-1}(s)}{\tau^{-1}(s)^{(2\gamma-1)(1+\delta_0)}} \ge c_1(\gamma)\Gamma(\varepsilon,\delta_0)^{2\gamma-1}\int_{\varepsilon/2}^{t} s^{((1/2)-\delta_0)(1-2\gamma)}\,ds.$$

If $\delta_0' = \delta_0(2\gamma-1)$, this in turn gives

$$\begin{aligned}\tau^{-1}(t)^{2-2\gamma-\delta_0'} &\ge c_1(\gamma)\Gamma(\varepsilon,\delta_0)^{2\gamma-1}\left[t^{1+(1/2-\delta_0)(1-2\gamma)} - \left(\frac{\varepsilon}{2}\right)^{1+(1/2-\delta_0)(1-2\gamma)}\right]\\ &\ge c_2(\gamma)\Gamma(\varepsilon,\delta_0)^{2\gamma-1}t^{(3/2)-\gamma+\delta_0'}.\end{aligned}$$

We have shown that if $\beta(\delta_0) = \frac{(3/2)-\gamma+\delta_0'}{2-2\gamma-\delta_0'}$, then for $\varepsilon \le t \le \rho_i \wedge \bar{\tau}_i$,

$$\begin{aligned}\tau^{-1}(t) &\ge c_2(\gamma)^{1/(2-2\gamma-\delta_0')}\Gamma(\varepsilon,\delta_0)^{(2\gamma-1)/(2-2\gamma-\delta_0')}t^{\beta(\delta_0)}\\ &\ge c_2(\gamma)^{1/(2-2\gamma-\delta_0')}(\Gamma(\varepsilon,\delta_0)\wedge 1)^2 t^{\beta(\delta_0)},\end{aligned}$$

where $\delta_0' < 1/4$ is used in the last line.

Recall the definition of the constant $\beta \in [1, \frac{3}{2})$ from (3.2). Use the above in (5.14) to see that there is a $c_3(\gamma) \in (0,1)$ so that for $\varepsilon \le t \le \rho_i \wedge \bar{\tau}_i \wedge 1$,

$$\begin{aligned}\bar{X}(t) &\ge [c_3(\gamma)(\Gamma(\varepsilon,\delta_0)\wedge 1)]^4 t^{\beta(\delta_0)(1+\delta_0)}\\ &> (2t)^{\beta+\delta_1},\end{aligned}$$

provided that $c_3(\gamma)(\Gamma(\varepsilon,\delta_0)\wedge 1) > 2t^{\delta_0}$, and δ_0 is chosen small enough depending on δ_1 and γ. By (5.13) we conclude that for $t \le 1$, and $\varepsilon \in (0,1]$,

$$\begin{aligned}&Q_i(\bar{X}_s \le (2s)^{\beta+\delta_1} \text{ for some } \varepsilon \le s \le \rho_i \wedge \bar{\tau}_i \wedge t)\\ &\quad\le Q_i(\Gamma(\varepsilon,\delta_0)\wedge 1 \le 2t^{\delta_0}/c_3(\gamma))\\ &\quad\le \eta_{\delta_0}(2t^{\delta_0}/c_3(\gamma)) + \mathbf{1}(2t^{\delta_0} \ge c_3(\gamma)) \equiv \eta_{4.1}(t).\end{aligned} \tag{5.15}$$

The above inequality is trivial for $t > 1$ as then the right-hand side is at least 1.

Next note that since $Z_t = \bar{X}(\tau_t)$ for $t \le T_Z$, $\bar{X}_u \equiv 1$ for $u \ge \bar{\tau}_i$, and $4Z$ has scale function $s(x) = -x^{-1}$ [see (V.48.5) in Rogers and Williams (1987)], we see that for $\varepsilon^{\delta_1} \le 2^{-\beta-\delta_1}$,

$$\begin{aligned}Q_i(\bar{X}_t \le (2\varepsilon)^{\beta+\delta_1} \text{ for some } t \ge 0) &\le Q_i(4Z \text{ hits } 4(2\varepsilon)^{\beta+\delta_1} \text{ before } 4)\\ &= \frac{s(4)-s(4\varepsilon)}{s(4)-s(4\cdot 2^{\beta+\delta_1}\varepsilon^{\beta+\delta_1})}\\ &= \frac{1-\varepsilon}{2^{-\beta-\delta_1}\varepsilon^{1-\beta-\delta_1}-\varepsilon}\\ &= \frac{1-\varepsilon}{2^{-\beta-\delta_1}\varepsilon^{-\delta_1}(\varepsilon^{1-\beta}-2^{\beta+\delta_1}\varepsilon^{\delta_1+1})}\end{aligned} \tag{5.16}$$

$$\leq \frac{1-\varepsilon}{2^{-\beta-\delta_1}\varepsilon^{-\delta_1}(\varepsilon^{1-\beta}-\varepsilon)}$$
$$\leq 2^{\beta+\delta_1}\varepsilon^{\delta_1} \leq 8\varepsilon^{\delta_1}.$$

The above bound is trivial if $\varepsilon^{\delta_1} > 2^{-\beta-\delta_1}$.

We combine (5.15) and (5.16) to conclude that

$$\begin{aligned}
&Q_i(\bar{X}_s \leq (s+\varepsilon)^{\beta+\delta_1} \text{ for some } 0 \leq s \leq \rho_i \wedge \bar{\tau}_i \wedge t) \\
&\quad \leq Q_i(\bar{X}_s \leq (2s)^{\beta+\delta_1} \text{ for some } \varepsilon \leq s \leq \rho_i \wedge \bar{\tau}_i \wedge t) \\
&\qquad + Q_i(\bar{X}_s \leq (2\varepsilon)^{\beta+\delta_1} \text{ for some } 0 \leq s \leq \varepsilon) \\
&\quad \leq \eta_{4.1}(t) + 8\varepsilon^{\delta_1}.
\end{aligned}$$

The result follows. $\square$

PROOF OF LEMMA 4.2. As in the previous proof we set

$$\bar{X}_t = \bar{U}_{s_i+(t\wedge\bar{\tau}_i)}(1) = \varepsilon + \bar{M}^i_{t\wedge\bar{\tau}_i}.$$

From (5.10) we have under Q_i

$$\bar{X}_t = \varepsilon + \bar{M}^{i,Q}_t + \int_0^{t\wedge\bar{\tau}_i} \bar{X}_s^{-1}\, d\langle \bar{M}^i\rangle_s, \tag{5.17}$$

where $\bar{M}^{i,Q}$ is an $(\mathcal{F}_{s_i+t})$-local martingale under Q_i. Therefore $\bar{X}$ is a bounded nonnegative submartingale under Q_i, and by the weak L^1 inequality

$$\begin{aligned}
Q_i(\bar{\tau}_i \leq t \wedge (T_R - s_i)^+) &= Q_i\Big(\sup_{s\leq t\wedge(T_R-s_i)^+} \bar{X}_s \geq 1\Big) \\
&\leq \int \bar{X}_{t\wedge(T_R-s_i)^+}\, dQ_i.
\end{aligned} \tag{5.18}$$

It is not hard to show that $\bar{M}^{i,Q}$ is actually a martingale under Q_i, but even without this we can localize and use Fatou's lemma to see that the right-hand side of (5.18) is at most

$$\varepsilon + E_{Q_i}\bigg[\int_0^t \mathbf{1}(s \leq (T_R - s_i)^+ \wedge \bar{\tau}_i)\bar{X}_s^{-1}\, d\langle \bar{M}^i\rangle_s\bigg] \equiv \varepsilon + I. \tag{5.19}$$

Next we use (2.6) and then the mean value theorem to see that

$$\begin{aligned}
I = E_{Q_i}\bigg[&\int_{s_i}^{s_i+t} \mathbf{1}(s \leq T_R \wedge (s_i + \bar{\tau}_i)) \\
&\times \int (U(s,x)^{2\gamma-1}U^i(s,x) + \bar{U}(s,x)^{2\gamma} - U(s,x)^{2\gamma})
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times \widetilde{U}^i(s,x)\widetilde{U}(s,x)^{-1}\,dx\,\bar{U}^i_s(1)^{-1}\,ds\bigg]\\
&\le \int_{s_i}^{s_i+t} E_{Q_i}\bigg[\mathbf{1}(s\le T_R\wedge(s_i+\bar{\tau}_i))\\
&\qquad\times\int (U(s,x)^{2\gamma-1}U^i(s,x)+2\gamma\bar{U}(s,x)^{2\gamma-1}\widetilde{U}^i(s,x))\,dx\\
&\qquad\qquad\times\bar{U}^i_s(1)^{-1}\bigg]\,ds\\
&\le 2\gamma R^{2\gamma-1}\int_{s_i}^{s_i+t} E_{Q_i}\bigg[\mathbf{1}(s\le s_i+\bar{\tau}_i)\int \bar{U}^i(s,x)\,dx\bar{U}^i_s(1)^{-1}\bigg]\,ds\\
&\le 2\gamma R^{2\gamma-1}t.
\end{aligned}
$$

We put the above bound into (5.19) and then use (5.18) to conclude that

$$Q_i(\bar{\tau}_i\le t\wedge(T_R-s_i)^+)\le\varepsilon+2\gamma R^{2\gamma-1}t$$

as required. □

Proof of Proposition 3.2. Fix $i\le N_\varepsilon$ and set

$$X_t=U^i_{s_i+(t\wedge\bar{\tau}_i)}(1),\qquad D_t=\widetilde{U}^i_{s_i+(t\wedge\bar{\tau}_i)}(1).$$

If $f(x,d)=d/(x+d)$, then

$$R_t\equiv\frac{\widetilde{U}^i_{s_i+(t\wedge\bar{\tau}_i)}(1)}{\bar{U}^i_{s_i+(t\wedge\bar{\tau}_i)}(1)}=f(X_t,D_t)\in[0,1]. \tag{5.20}$$

Proposition 2.1 shows that X and D are right-continuous semimartingales with left limits. We will work under Q_i so that the denominator of R is strictly positive for all $t\ge 0$ Q_i-a.s. Our goal will be to show that R remains small on $[0,t\wedge v_i]$ for t small with high probability, uniformly in ε. Then $U^i_{s_i+s}(1)$ will be bounded below by a constant times $\bar{U}_{s_i+s}(1)$ on this interval with high probability, and the latter satisfies a uniform escape rate on the interval by the definition of v_i.

From Proposition 2.1, and in particular (2.4) and (2.5), we have

$$\widetilde{U}^i_{s_i+(t\wedge\bar{\tau}_i)}(1)=\widetilde{M}^i_t+K^{i,U}_{s_i+(t\wedge\bar{\tau}_i)}(1),$$

where $\widetilde{M}^i$ is the continuous $(\mathcal{F}_{s_i+t})$-local martingale (under P) given by

$$\widetilde{M}^i_t=\int_{s_i}^{s_i+(t\wedge\bar{\tau}_i)}(\bar{U}(s,x)^{2\gamma}-U(s,x)^{2\gamma})^{1/2}\sqrt{\frac{\widetilde{U}^i(s,x)}{\widetilde{U}(s,x)}}\widetilde{W}^{i,U}(ds,dx),$$

and $K^{i,U}_{s_i+\cdot}$ is a right-continuous nondecreasing process. By Girsanov's theorem [Theorem VIII.1.4 in Revuz and Yor (1999)] there is a continuous $(\mathcal{F}_{s_i+t})$-local martingale under Q_i, $\widetilde{M}^{i,Q}$, so that

$$\begin{aligned}\widetilde{M}^i_t &= \widetilde{M}^{i,Q}_t + \int_{s_i}^{s_i+(t\wedge\bar{\tau}_i)} \bar{U}^i_s(1)^{-1}\,d\langle\widetilde{M}^i,\bar{M}^i\rangle_s\\ &= \widetilde{M}^{i,Q}_t\\ &\quad + \int_{s_i}^{s_i+(t\wedge\bar{\tau}_i)}\int(\bar{U}(s,x)^{2\gamma}-U(s,x)^{2\gamma})\frac{\widetilde{U}^i(s,x)\widetilde{U}(s,x)^{-1}}{\bar{U}^i_s(1)}\,dx\,ds.\end{aligned}\tag{5.21}$$

From (2.1) we have

$$U^i_{s_i+(t\wedge\bar{\tau}_i)}(1)=\varepsilon+M^i_t-K^{i,U}_{s_i+(t\wedge\bar{\tau}_i)}(1),$$

where M^i is the continuous $(\mathcal{F}_{s_i+t})$-local martingale (under P),

$$M^i_t=\int_{s_i}^{s_i+(t\wedge\bar{\tau}_i)}\int U(s,x)^{\gamma-(1/2)}U^i(s,x)^{1/2}W^{i,U}(ds,dx).$$

Another application of Girsanov's theorem implies there is a continuous $(\mathcal{F}_{s_i+t})$-local martingale under Q_i, $M^{i,Q}_t$, such that

$$M^i_t=M^{i,Q}_t+\int_{s_i}^{s_i+(t\wedge\bar{\tau}_i)}\int\frac{U(s,x)^{2\gamma-1}U^i(s,x)}{\bar{U}^i_s(1)}\,dx\,ds.\tag{5.22}$$

Note that $\langle M^i,\widetilde{M}^i\rangle=0$ and so $M^{i,Q}$ and $\widetilde{M}^{i,Q}$ are also orthogonal under Q_i.

If

$$J_t=\sum_{s\le t}f(X_s,D_s)-f(X_{s-},D_{s-})-f_x(X_{s-},D_{s-})\Delta X_s-f_d(X_{s-},D_{s-})\Delta D_s,$$

then Itô's lemma [e.g., Theorem VI.39.1 in Rogers and Williams (1987)] shows that under Q_i,

$$\begin{aligned}R_t &= R_0+\int_0^t f_x(X_{s-},D_{s-})\,dX_s+\int_0^t f_d(X_{s-},D_{s-})\,dD_s\\ &\quad+\int_0^{t\wedge\bar{\tau}_i}\frac{1}{2}f_{xx}(X_{s-},D_{s-})\int U(s_i+s,x)^{2\gamma-1}U^i(s_i+s,x)\,dx\,ds\\ &\quad+\int_0^{t\wedge\bar{\tau}_i}\frac{1}{2}f_{dd}(X_{s-},D_{s-})\int[\bar{U}(s_i+s,x)^{2\gamma}-U(s_i+s,x)^{2\gamma}]\\ &\qquad\times\widetilde{U}^i(s_i+s,x)\widetilde{U}(s_i+s,x)^{-1}\,dx\,ds+J_t.\end{aligned}\tag{5.23}$$

Since

$$\Delta X_t=-\Delta K^{i,U}_{s_i+(t\wedge\bar{\tau}_i)}(1)=-\Delta D_t,$$

and $f_x = -d(x+d)^{-2}$, $f_d = x(x+d)^{-2}$, we conclude that

$$\begin{aligned} J_t &= \sum_{s\le t}[f(X_{s-}-\Delta D_s, D_{s-}+\Delta D_s) - f(X_{s-}, D_{s-}) \\ &\qquad + [f_x - f_d](X_{s-}, D_{s-})\Delta D_s] \\ &= \sum_{s\le t}\frac{\Delta D_s}{X_{s-}+D_{s-}} - \frac{\Delta D_s}{X_{s-}+D_{s-}} = 0. \end{aligned}$$

We use $f_{xx} = 2d(x+d)^{-3}$, $f_{dd} = -2x(x+d)^{-3}$, (5.21) and (5.22) in (5.23) to conclude that if $\bar{X}_t = \bar{U}^i_{s_i+(t\wedge\bar{\tau}_i)}(1)$ and

$$N_t = \int_0^t -D_{s-}\bar{X}_s^{-2}\,dM_s^{i,Q} + \int_0^t X_{s-}\bar{X}_s^{-2}\,d\widetilde{M}_s^{i,Q},$$

then

$$\begin{aligned} R_t &= R_0 + N_t + \int_0^{t\wedge\bar{\tau}_i}(-D_{s-}\bar{X}_s^{-3})\int U(s_i+s,x)^{2\gamma-1}U^i(s_i+s,x)\,dx\,ds \\ &\quad + \int_0^{t\wedge\bar{\tau}_i} D_{s-}\bar{X}_s^{-2}\,dK^{i,U}_{s_i+s}(1) \\ &\quad + \int_0^{t\wedge\bar{\tau}_i} X_s\bar{X}_s^{-3}\int[\bar{U}(s_i+s,x)^{2\gamma} - U(s_i+s,x)^{2\gamma}] \\ &\qquad\qquad \times \bar{U}^i(s_i+s,x)\widetilde{U}(s_i+s,x)^{-1}\,dx\,ds \\ &\quad + \int_0^{t\wedge\bar{\tau}_i} X_{s-}\bar{X}_s^{-2}\,dK^{i,U}_{s_i+s}(1) \\ &\quad + \int_0^{t\wedge\bar{\tau}_i} D_{s-}\bar{X}_s^{-3}\int U(s_i+s,x)^{2\gamma-1}U^i(s_i+s,x)\,dx\,ds \\ &\quad - \int_0^{t\wedge\bar{\tau}_i} X_s\bar{X}_s^{-3}\int[\bar{U}(s_i+s,x)^{2\gamma} - U(s_i+s,x)^{2\gamma}] \\ &\qquad\qquad \times \widetilde{U}^i(s_i+s,x)\widetilde{U}(s_i+s,x)^{-1}\,dx\,ds \\ &= R_0 + N_t + \int_0^{t\wedge\bar{\tau}_i}\bar{X}_s^{-1}\,dK^{i,U}_{s_i+s}(1). \end{aligned} \tag{5.24}$$

Under Q_i, N is a continuous $(\mathcal{F}_{s_i+t})$-local martingale, and the last term in (5.24) is nondecreasing. It follows from this and $R\in[0,1]$ that

$$R \text{ is an } (\mathcal{F}_{s_i+t})\text{-submartingale under } Q_i. \tag{5.25}$$

As $R_0 = K^{i,U}_{s_i}(1)/\varepsilon$, integration by parts shows that

$$R_t = R_0 + N_t + \frac{K^{i,U}_{s_i+(t\wedge\bar{\tau}_i)}(1)}{\bar{X}_t} - \frac{K^{i,U}_{s_i}(1)}{\varepsilon} - \int_0^t K^{i,U}_{s_i+s}(1)\,d\left(\frac{1}{\bar{X}_s}\right)$$

$$= N_t - \int_0^t K^{i,U}_{s_i+s}(1)\, d\biggl(\frac{1}{\bar X_s}\biggr) + \frac{K^{i,U}_{s_i+(t\wedge\bar\tau_i)}(1)}{\bar U^i_{s_i+(t\wedge\bar\tau_i)}(1)}. \tag{5.26}$$

Another application of Itô's lemma using (5.1) and (5.10) shows that

$$\begin{aligned}\bar X_t^{-1} &= \varepsilon^{-1} - \int_0^t \bar X_s^{-2}\, d\bar X_s + \int_0^{t\wedge\bar\tau_i} \bar X_s^{-3}\, d\langle \bar M^i\rangle_s \\ &= \varepsilon^{-1} - \int_0^t \bar X_s^{-2}\, d\bar M^{i,Q}_s - \int_0^{t\wedge\bar\tau_i} \bar X_s^{-3}\, d\langle \bar M^i\rangle_s + \int_0^{t\wedge\bar\tau_i} \bar X_s^{-3}\, d\langle \bar M^i\rangle_s \\ &= \varepsilon^{-1} - \int_0^t \bar X_s^{-2}\, d\bar M^{i,Q}_s.\end{aligned}$$

Therefore $\bar X_t^{-1}$ is a continuous $(\mathcal{F}_{s_i+t})$-local martingale under Q_i and hence the same is true of $N_t^R = N_t - \int_0^t K^{i,U}_{s_i+s}(1)\, d(\frac{1}{\bar X_s})$. From (5.26) we have

$$R_t = N_t^R + \frac{K^{i,U}_{s_i+(t\wedge\bar\tau_i)}(1)}{\bar U^i_{s_i+(t\wedge\bar\tau_i)}(1)}. \tag{5.27}$$

Recall from (2.8) and (2.2) that

$$\Delta K^{i,U}_{s_i+t}(1) \le \varepsilon \qquad \text{for all } t \ge 0. \tag{5.28}$$

Assume that (recall $\beta < 3/2$)

$$0 < 2\delta_0 \le \delta_1 \le \tfrac14(\tfrac32 - \beta) \equiv \delta_{3.2}(\gamma). \tag{5.29}$$

These last two inequalities (which give $\frac32 - \beta - \delta_1 - 2\delta_0 > 0$) together with the continuity of $\bar U^i_{s_i+\cdot}(1)$ [recall Proposition 2.1(a)], and the definitions of $\theta_i \ge v_i$ and $H_i \ge v_i$ imply that

$$\sup_{s\le v_i\wedge t} \frac{K^{i,U}_{s_i+s}(1)}{\bar U^i_{s_i+s}(1)} \le \sup_{s\le v_i\wedge t} \frac{(s+\varepsilon)^{(3/2)-2\delta_0} + \varepsilon}{(s+\varepsilon)^{\beta+\delta_1}} \le (t+\varepsilon)^{(3/2)-\beta-2\delta_0-\delta_1} + \varepsilon^{1-\beta-\delta_1},$$

and so from (5.27)

$$\sup_{s\le v_i\wedge t} |N_s^R| \le 1 + (t+\varepsilon)^{(3/2)-\beta-2\delta_0-\delta_1} + \varepsilon^{1-\beta-\delta_1} < \infty. \tag{5.30}$$

We now apply the weak L^1 inequality to the nonnegative submartingale R [recall (5.25)] to conclude that ($\sup\varnothing = 0$)

$$\begin{aligned}&Q_i\Bigl(\sup_{\varepsilon^{2/3}\le s\le v_i\wedge t} R_s \ge 1/2\Bigr) \\ &\quad = E_{Q_i}\biggl[Q_i\biggl(\sup_{\varepsilon^{2/3}\le s\le v_i\wedge t} R_s \ge \frac12 \bigg| \mathcal{F}_{\varepsilon^{2/3}}\biggr)\mathbf{1}(v_i\wedge t \ge \varepsilon^{2/3})\biggr]\end{aligned}$$

$$
\begin{aligned}
&\le 2E_{Q_i}[R_{v_i\wedge t}\mathbf{1}(v_i\wedge t\ge \varepsilon^{2/3})]\mathbf{1}(t\ge\varepsilon^{2/3}) \\
&\le 2E_{Q_i}\left[R_{(v_i\wedge t)-}+\frac{\Delta K^{i,U}_{s_i+(v_i\wedge t)}(1)}{\bar U^i_{s_i+(v_i\wedge t)}(1)}\mathbf{1}(v_i\wedge t\ge\varepsilon^{2/3})\right]\mathbf{1}(t\ge\varepsilon^{2/3}).
\end{aligned}
\tag{5.31}
$$

By (5.28) and the definition of $H_i\ge v_i$ we have

$$
\begin{aligned}
&\frac{\Delta K^{i,U}_{s_i+(v_i\wedge t)}(1)}{\bar U^i_{s_i+(v_i\wedge t)}(1)}\mathbf{1}(v_i\wedge t\ge\varepsilon^{2/3}) \\
&\qquad\le \frac{\varepsilon}{(\varepsilon+v_i\wedge t)^{\beta+\delta_1}}\mathbf{1}(v_i\wedge t\ge\varepsilon^{2/3}) \\
&\qquad\le \varepsilon^{1-(2/3)(\beta+\delta_1)}.
\end{aligned}
\tag{5.32}
$$

From (5.27) and the definitions of $H_i\ge v_i$ and $\theta_i\ge v_i$ we have

$$
\begin{aligned}
E_{Q_i}[R_{(v_i\wedge t)-}] &= E_{Q_i}[N^R_{v_i\wedge t}]+E_{Q_i}[K^{i,U}_{s_i+(v_i\wedge t)-}(1)/\bar U^i_{s_i+(v_i\wedge t)}(1)] \\
&\le E_{Q_i}[(\varepsilon+(v_i\wedge t))^{(3/2)-\beta-2\delta_0-\delta_1}] \\
&\le (\varepsilon+t)^{(3/2)-\beta-2\delta_0-\delta_1},
\end{aligned}
\tag{5.33}
$$

where we used (5.30) to see that $N^R_{v_i\wedge t}$ is a mean zero martingale and also applied (5.29) to see the exponent is positive. Inserting (5.32) and (5.33) into (5.31) and using (5.29), we get for $t\le 1$,

$$
\begin{aligned}
&Q_i\left(\sup_{\varepsilon^{2/3}\le s\le v_i\wedge t} R_s\ge\frac12\right) \\
&\qquad\le [(\varepsilon+t)^{(3/2)-\beta-2\delta_0-\delta_1}+\varepsilon^{1-(2/3)(\beta+\delta_1)}]\mathbf{1}(t\ge\varepsilon^{2/3}) \\
&\qquad\le 2^{3/2}t^{(3/2)-\beta-2\delta_1}+t^{(3/2)-(\beta+\delta_1)}\le 5t^{(3/2)-\beta-2\delta_1}.
\end{aligned}
\tag{5.34}
$$

Equation (5.29) implies $(3/2)-\beta-2\delta_1\ge(1/2)((3/2)-\beta)$, and so for $t\le 1$ we conclude

$$
Q_i\left(\sup_{\varepsilon^{2/3}\le s\le v_i\wedge t} R_s\ge\frac12\right)\le 5t^{(1/2)((3/2)-\beta)}.
$$

The above is trivial for $t>1$. On $\{\sup_{\varepsilon^{2/3}\le s\le v_i\wedge t}R_s<1/2\}$ we have for all $s\in[\varepsilon^{2/3},t\wedge v_i]$,

$$
U^i_{s_i+s}(1)\ge\tfrac12\bar U^i_{s_i+s}(1)\ge\tfrac12 s^{\beta+\delta_1},
$$

and so $B_i(t\wedge v_i)$ occurs. The result follows with $p_{3.2}=\frac12(\frac32-\beta)\in(0,\frac14]$ (as $\gamma\ge 1/2$). $\square$

6. Propagation speed of the supports and a comparison principle: Proofs of Lemmas 4.4 and 4.5. If $a>0$, $1>\gamma\ge 1/2$ and $X_0\in C^+_{\mathrm{rap}}$, then Theorems 2.5 and 2.6 of Shiga (1994) show the existence of continuous C^+_{rap}-valued solutions to

$$\frac{\partial X}{\partial t}=\frac{1}{2}\Delta X+aX^{\gamma}\dot W, \tag{6.1}$$

where as usual $\dot W$ is a space–time white noise on $\mathbb{R}_+\times\mathbb{R}$. Theorem 1.1 of Mytnik (1998) then shows the laws $\{P_{X_0}:X_0\in C^+_{\mathrm{rap}}\}$ of these processes on $C(\mathbb{R}_+,C^+_{\mathrm{rap}})$ are unique.

We start with a quantified version of Theorem 3.5 of Mueller and Perkins (1992) applied to the particular equation (6.1).

Lemma 6.1. *Assume X satisfies (6.1) with $X_0=J^{x_0}_{\varepsilon}$ for $x_0\in\mathbb{R}$ and $\varepsilon\in(0,1]$. If $\gamma\in(1/2,3/4)$ choose $\delta=\delta(\gamma)\in(0,1/5)$ sufficiently small so that $\beta_0=\beta_0(\gamma)=\frac{2\gamma-\delta}{1-\delta}\in(1,3/2)$ and for $N>1$, define*

$$T_N=\inf\left\{t\ge 0:\int X(t,x)^{\delta}\,dx\ge N\right\}.$$

If $\gamma=1/2$, set $\beta_0=1$ and $T_N=\infty$. For $\delta_0\in(0,1/4]$, define

$$\rho=\inf\{t\ge 0:S(X_t)\not\subset[x_0-\varepsilon^{1/2}-t^{(1/2)-\delta_0},x_0+\varepsilon^{1/2}+t^{(1/2)-\delta_0}]\}. \tag{6.2}$$

There is a $c_{6.1}>0$ (depending on γ) so that

$$P(\rho\le t\wedge T_N)\le c_{6.1}a^{-1}N^{\beta_0-1}\varepsilon\exp(-t^{-\delta_0}/c_{6.1})\qquad\text{for all } \varepsilon,t\in(0,1].$$

Proof. Since X is unique in law, the construction in Section 4 of Mueller and Perkins (1992) allows us to assume the existence of a historical process H_t, a continuous $M_F(C)$-valued process, associated with X. Here C is the space of continuous $\mathbb{R}$-valued paths. H will satisfy the martingale problem (M_{X_0}) in Mueller and Perkins (1992), and the relationship with X is that

$$H_t(\{y\in C:y_t\in B\})=X_t(B)\quad\text{for all } t\ge 0 \text{ and Borel subsets } B \text{ of } \mathbb{R}. \tag{6.3}$$

Hence the hypotheses of Theorem 3.5 of Mueller and Perkins (1992) are satisfied with $a_k\equiv a$ for all k. If $I_t=[x_0-\sqrt{\varepsilon}-t^{(1/2)-\delta_0},x_0+\sqrt{\varepsilon}+t^{(1/2)-\delta_0}]$, that result implies $S(X_t)\subset I_t$ for small enough t a.s., but we need to quantify this inclusion and so will follow the proof given there, pointing out some minor changes and simplifications as we go.

If $\gamma=1/2$, X is the density of one-dimensional super-Brownian motion, and the argument in Mueller and Perkins (1992) and its quantification are both much easier. As a result we will assume $3/4 > \gamma > 1/2$ in what follows and leave the simpler case $\gamma = 1/2$ for the reader. The fact that $a_k = a$ for all k [i.e., for us $a(u) = au^\gamma$ for all u in the notation of Mueller and Perkins (1992)], means that in the localization in Mueller and Perkins (1992), the times $\{T_N\}$ may be chosen to agree with our definition of T_N. We will work with the cruder modulus of continuity, $\psi(t) = \frac{1}{2}t^{(1/2)-\delta_0}$, in place of the more delicate $ch(t) = c(t\log^+(1/t))^{1/2}$ in Mueller and Perkins (1992), leading to better bounds.

If

$$G_{n,j,k} = \{y \in C : |y(k2^{-n}) - y(j2^{-n})| > \psi((k-j)2^{-n})\}, \qquad 0 \le j < k; j,k,n \in \mathbb{Z}_+,$$

and B is a standard one-dimensional Brownian motion, then for $k - j \le 2^{n/2}$, (3.16) of Mueller and Perkins (1992) becomes

$$\begin{aligned} &Q_{X_0}(H_{(k+1)2^{-n}}(G_{n,j,k}) > 0, T_N \ge (k+1)2^{-n}) \\ &\quad \le c_1 N^{\beta_0 - 1} a^{-1} 2^n X_0(1) P(|B(k2^{-n}) - B(j2^{-n})| > \psi((k-j)2^{-n}))^{2-\beta_0} \\ &\quad \le c_2 N^{\beta_0 - 1} a^{-1} 2^n \varepsilon \exp(-\tfrac{1}{16} 2^{n\delta_0}) \qquad (\text{recall } \beta_0 < 3/2). \end{aligned}$$

Now we sum the above bound over $0 \le j < k \le 2^n$, $k - j \le 2^{n/2}$, $n \ge m$ and argue as in the proof of Theorem 3.5 in Mueller and Perkins (1992) to see that if

$$\eta_m = c_3 N^{\beta_0 - 1} a^{-1} \varepsilon \exp(-2^{(m\delta_0/2) - 4}),$$

then with probability at least $1 - \eta_m$,

$$H_t(G_{n,j,k}) = 0 \qquad \text{for all } 0 \le j < k \le 2^n, k - j \le 2^{n/2}, (k+1)2^{-n} \le T_N, \quad t \ge (k+1)2^{-n}, \text{ and } n \ge m.$$

Rearranging this as in the proof of Theorem 3.5 of Mueller and Perkins (1992), we have with probability at least $1 - \eta_m$,

$$\begin{aligned} &|y(k2^{-n}) - y(j2^{-n})| \le \psi((k-j)2^{-n}) \qquad \text{for all } 0 \le j < k, k - j \le 2^{n/2}, \\ &(k+1)2^{-n} \le t \text{ and } n \ge m \text{ for } H_t\text{-a.a. } y \text{ for all } t \le T_N \wedge 1. \end{aligned} \tag{6.4}$$

Next, we can argue as in the last part of the proof of Mueller and Perkins (1992), which was a slightly modified version of Lévy's classical derivation of the exact Brownian modulus of continuity, to see that (6.4) implies

$$|y(v) - y(u)| \le 2\psi(|v-u|) \qquad \text{for all } 0 \le u < v \le t \text{ satisfying } |v-u| \le 2^{-m/2} \quad \text{for } H_t\text{-a.a. } y \text{ for all } t \le T_N \wedge 1.$$

In particular, the above implies that

$$P(|y(t)-y(0)|\le 2\psi(t)\ H_t\text{-a.a. } y \text{ for all } t\le 2^{-m/2}\wedge T_N)\ge 1-\eta_m.$$

Now $H_t(|y(0)-x_0|>\sqrt{\varepsilon})$ is a nonnegative martingale starting at 0 by the martingale problem for H [just as in the proof of Corollary 3.9 in Mueller and Perkins (1992)] and so is identically 0 for all t a.s. Therefore, the above and (6.3) imply that

$$P(\rho < 2^{-m/2}\wedge T_N)\le \eta_m.$$

A simple interpolation argument now gives the required bound. □

COROLLARY 6.2. *Assume X, δ_0 and ρ are as in Lemma 6.1. There is a $c_{6.2}>0$, depending on a, δ_0 and γ, so that*

$$P(\rho\le t)\le c_{6.2}\varepsilon(t\vee\varepsilon)\qquad \textit{for all } t,\varepsilon\in(0,1].$$

PROOF. We clearly may assume $x_0=0$ by translation invariance. By Lemma 6.1 with $N=N_0\equiv 8$ and β_0, T_{N_0} as in that result, we have

$$P(\rho\le t)\le c_{6.1}a^{-1}8^{\beta_0-1}\varepsilon\exp(-t^{-\delta_0}/c_{6.1})+P(t\wedge T_{N_0}<\rho\le t). \tag{6.5}$$

The result is now immediate if $\gamma=1/2$, so we assume $\gamma\in(1/2,3/4)$. If $\delta\in(0,\frac{1}{5})$ is as in Lemma 6.1, $I_s=[-\sqrt{\varepsilon}-s^{(1/2)-\delta_0},\sqrt{\varepsilon}+s^{(1/2)-\delta_0}]$, and $0<t\le 1$, then

$$\begin{aligned}
&P(t\wedge T_{N_0}<\rho\le t)\\
&\quad\le P(T_{N_0}<t\wedge\rho)\\
&\quad\le P\biggl(\int_{I_s}X(s,x)^\delta\,dx>8 \text{ for some } s\le t\wedge\rho\biggr)\\
&\quad\le P\biggl(\biggl(\int X(s,x)\,dx\biggr)^\delta|I_s|^{1-\delta}>8 \text{ for some } s\le t\biggr)\\
&\quad\le P\Bigl(\sup_{s\le t}X_s(1)>\lambda\Bigr),
\end{aligned}\tag{6.6}$$

where $\lambda=8^{1/\delta}[[2(\sqrt{\varepsilon}+t^{(1/2)-\delta_0})]^{(1-\delta)/\delta}]^{-1}$. Recall that $X_t(1)$ is a continuous nonnegative local martingale starting at ε, and so by the weak L^1 inequality and Fatou's lemma the right-hand side of (6.6) is at most

$$\begin{aligned}
\lambda^{-1}E[X_0(1)]&\le \varepsilon 2^{-1-(2/\delta)}(\sqrt{\varepsilon}+t^{1/4})^{(1-\delta)/\delta}\qquad(\text{by } \delta_0\le 1/4)\\
&\le \varepsilon[\max(t,\varepsilon^2)]^{(1-\delta)/(4\delta)}\\
&\le \varepsilon\max(t,\varepsilon)\qquad(\text{since } \delta<1/5).
\end{aligned}$$

We use the above bound in (6.5) to conclude that

$$\begin{aligned} P(\rho \le t) &\le [c_{6.1} a^{-1} 8^{\beta_0 - 1} \exp(-t^{-\delta_0}/c_{6.1}) + (t \vee \varepsilon)]\varepsilon \\ &\le c_{6.2}(t \vee \varepsilon)\varepsilon. \end{aligned}$$

□

The next proposition will allow us to extend the above bound to a larger class of SPDEs. It will be proved at the end of this section.

PROPOSITION 6.3. *Let $a > 0$, $1 > \gamma \ge 1/2$ and Z be a continuous C^+_{rap}-valued solution to the following SPDE:*

$$\frac{\partial Z}{\partial t} = \frac{1}{2}\Delta Z + \sigma(Z_s, s, \omega)\dot{W}^1, \tag{6.7}$$

where $\dot{W}^1$ is a space time white noise, σ is Borel $\times$ previsable, and

$$\sigma(y, s, \omega) \ge a y^\gamma \qquad \forall s, y, P\text{-a.s. } \omega.$$

Assume also for each $t > 0$ we have

$$\sup_{s \le t, x \in \mathbb{R}} E[Z(s,x)^2] < \infty. \tag{6.8}$$

Let X be a continuous C^+_{rap}-valued solution to the following SPDE, perhaps on a different space,

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + a X^\gamma \dot{W}, \tag{6.9}$$

with $Z(0,\cdot) = X(0,\cdot) \in C^+_{\mathrm{rap}}$. Let A be a Borel set in $\mathbb{R}_+ \times \mathbb{R}$. Then

$$P(\mathrm{supp}(Z) \cap A = \varnothing) \ge P_{X_0}(\mathrm{supp}(X) \cap A = \varnothing).$$

We will apply this result with $Z(t,x) = \bar{U}^i(s_i + t, x)$. To ensure (6.8) we will need the following moment bound which will also give Lemma 4.5. It will be proved in Appendix A.

LEMMA 6.4. *For any $q, T > 0$, there exists $C_{q,T}$ such that*

$$\sup_{0 < \varepsilon \le 1} E\Big[\sup_{s \le T, x \in \mathbb{R}} (\bar{U}(s,x)^q + \bar{V}(s,x)^q)\Big] \le C_{q,T}. \tag{6.10}$$

The proof of the above lemma is based on a simple adaptation of the methods used for the proof of Proposition 1.8(a) of Mytnik, Perkins and Sturm (2006), and in particular Lemma A.3 of that paper.

PROOF OF LEMMA 4.5. This result with $\delta_{4.5}(t) = C_{1/2,2} t^{\varepsilon_0/2}$ is an immediate corollary of Markov's lemma and the above lemma with $q = 1/2$. □

PROOF OF LEMMA 4.4. We first fix $1 \le i \le N_\varepsilon$ and argue conditionally on $\mathcal{F}_{s_i}$. Note that the inequalities in (5.3) hold pointwise, that is, without integrating over space. These inequalities together with (2.6), Lemma 6.4 and Proposition 2.1 show the hypotheses of Proposition 6.3 hold with $Z(t,x) = \bar{U}^i(s_i+t,x)$, $Z_0 = J_\varepsilon^{x_i}$ and $a = 2^{1/2-\gamma}$. We apply this result to the open set

$$A = A_t = \{(s,y) : |y - x_i| > \varepsilon^{1/2} + s^{(1/2)-\delta_0}, 0 < s < t\}$$

and conclude that if ρ is as in Lemma 6.1, then

$$P(\rho_i < t) = P_{J_\varepsilon^{x_i}}(\mathrm{supp}(Z) \cap A \neq \varnothing) \le P(\rho < t).$$

Corollary 6.2 now shows there is a $c_{4.4} = c_{4.4}(\gamma, \delta_0)$ so that for $\varepsilon, t \in (0,1]$,

$$P(\rho_i \le t) \le c_{4.4}\varepsilon(t \vee \varepsilon).$$

It follows that for $p, \varepsilon, t \in (0,1]$,

$$P\left(\bigcup_{i=1}^{pN_\varepsilon}\{\rho_i \le t\}\right) \le \sum_{i=1}^{\lfloor pN_\varepsilon \rfloor} P(\rho_i \le t) \le c_{4.4}\lfloor pN_\varepsilon\rfloor \varepsilon(t \vee \varepsilon) \le c_{4.4} p(t \vee \varepsilon)\mathbf{1}(p \ge \varepsilon).$$

This finishes the proof of Lemma 4.4. $\square$

We next turn to the proof of Proposition 6.3. Recall from the discussion at the beginning of this section that for each $X_0 \in C_{\mathrm{rap}}^+$ there is a unique law P_{X_0} on $C(\mathbb{R}_+, C_{\mathrm{rap}}^+)$ of the solution to (6.9). We assume the hypotheses of Proposition 6.3 for the rest of this section.

LEMMA 6.5. *Let $\gamma \in [1/2, 1)$. For any nonnegative $\phi \in L^1(\mathbb{R})$, and $t, s \ge 0$, there exists a sequence of $M_F(\mathbb{R})$-valued processes $\{Y^n\}_{n \ge 0}$ such that $Y_0^n(dx) = \phi(x)\,dx$ and*

$$E[e^{-\langle \phi, Z_t\rangle} | \mathcal{F}_s^Z] \ge E[e^{-\langle \phi, X_{t-s}\rangle} | X_0 = Z_s] \tag{6.11}$$

$$= \lim_{n\to\infty} E_\phi^{Y^n}[e^{-\langle Y_{t-s}^n, Z_s\rangle}], \tag{6.12}$$

where $P_\phi^{Y^n}$ is the probability law of Y^n.

PROOF. We may assume without loss of generality that $a = 1$, as only trivial adjustments are needed to the handle general $a > 0$. First we will prove the lemma for $\gamma > 1/2$ and then explain the modifications for the $\gamma = 1/2$ case. For $\gamma \in (1/2, 1)$, (6.12) follows from Proposition 2.3 of Mytnik (1998). To simplify the exposition let us take $s = 0$. For $s > 0$ the proof goes along the same lines as it depends only on the martingale properties of Z.

By the proof of Lemma 3.3 in Mytnik (1998) we get that for each n there exists a stopping time $\tilde{\gamma}_k(t) \le t$ and an $M_F(\mathbb{R})$-valued process Y^n such that, for $\eta = \frac{2\gamma(2\gamma-1)}{\Gamma(2-2\gamma)}$, and

$$g(u,y) = \int_0^u (e^{-\lambda y} - 1 + \lambda y)\lambda^{-2\gamma-1}\, d\lambda, \qquad u, y \ge 0,$$

we have

$$\begin{aligned}
&E[e^{-\langle Y^n_{\tilde{\gamma}_k(t)}, Z_{t-\tilde{\gamma}_k(t)}\rangle} | Y^n_0 = \phi] \\
&\quad = E_\phi[e^{-\langle \phi, Z_t\rangle}] \\
&\qquad - \frac{1}{2} E\biggl[\int_0^{\tilde{\gamma}_k(t)} e^{-\langle Y^n_s, Z_{t-s}\rangle} \biggl\{\eta \int_{\mathbb{R}} (Y^n_s(x))^2 g(1/n, Z_{t-s}(x))\, dx \\
&\qquad\qquad + \langle \sigma(Z_{t-s})^2 - (Z_{t-s})^{2\gamma}, (Y^n_s)^2\rangle\biggr\}\, ds\biggr] \qquad (6.13)\\
&\quad \le E_\phi[e^{-\langle \phi, Z_t\rangle}] \\
&\qquad - \frac{1}{2} E\biggl[\int_0^{\tilde{\gamma}_k(t)} e^{-\langle Y^n_s, Z_{t-s}\rangle} \eta \int_{\mathbb{R}} (Y^n_s(x))^2 g(1/n, Z_{t-s}(x))\, dx\, ds\biggr].
\end{aligned}$$

If $k = k_n = \ln(n)$, we can easily get [as in the proof of Lemma 3.4 of Mytnik (1998)] that

$$\begin{aligned}
&E\biggl[\int_0^{\tilde{\gamma}_{k_n}(t)} e^{-\langle Y^n_s, Z_{t-s}\rangle} \eta \int_{\mathbb{R}} (Y^n_s(x))^2 g(1/n, Z_{t-s}(x))\, dx\, ds\biggr] \\
&\quad \le C \sup_{x, s \le t} E[Z_s(x)^2] k_n n^{2\gamma-2} \qquad (6.14)\\
&\quad \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Here we used (6.8) in the last line. Moreover, as is shown in the proof of Lemma 3.5 of Mytnik (1998), we have

$$P(\tilde{\gamma}_{k_n}(t) < t) \to 0 \qquad \text{as } n \to \infty,$$

or equivalently,

$$P(\tilde{\gamma}_{k_n}(t) = t) \to 1 \qquad \text{as } n \to \infty.$$

Hence we get from (6.13), (6.14) and the above

$$\begin{aligned}
&\lim_{n\to\infty} E[e^{-\langle Y^n_t, Z_0\rangle} | Y^n_0 = \phi] \\
&\quad = \lim_{n\to\infty} E[e^{-\langle Y^n_{\tilde{\gamma}_{k_n}(t)}, Z_{t-\tilde{\gamma}_{k_n}(t)}\rangle} | Y^n_0 = \phi] \\
&\quad \le E[e^{-\langle \phi, Z_t\rangle}] \qquad \forall t \ge 0.
\end{aligned}$$

But by Lemma 3.5 of Mytnik (1998) we have

$$\lim_{n\to\infty} E[e^{-\langle Y_t^n, Z_0\rangle} | Y_0^n = \phi] = E[e^{-\langle \phi, X_t\rangle}] \qquad \forall t \geq 0 \tag{6.15}$$

and we are done for $\gamma \in (1/2, 1)$.

The case $\gamma = 1/2$ is even easier. Now X is just a super-Brownian motion. Now take $Y^n = Y$ for all n, where Y is a solution to the log-Laplace equation

$$\frac{\partial Y_t}{\partial t} = \frac{1}{2}\Delta Y_t - \frac{1}{2}(Y_t)^2,$$

so that (6.15) is the standard exponential duality for super-Brownian motion. Then (6.13) follows with $\tilde{\gamma}_k(t) = t$, and $\eta = 0$, and so the result follows immediately for $\gamma = 1/2$. □

LEMMA 6.6. *For any $k \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_k$ and $\phi_1, \ldots, \phi_k \geq 0$,*

$$E[e^{-\sum_{i=1}^k \langle \phi_i, Z_{t_i}\rangle}] \geq E[e^{-\sum_{i=1}^k \langle \phi_i, X_{t_i}\rangle}]. \tag{6.16}$$

PROOF. The proof goes by induction. For $k = 1$ it follows from the previous lemma. Suppose the equality holds for $k - 1$. Let us check it for k:

$$\begin{aligned}
&E[e^{-\sum_{i=1}^k \langle \phi_i, Z_{t_i}\rangle}] \\
&\quad = E[e^{-\sum_{i=1}^{k-1} \langle \phi_i, Z_{t_i}\rangle} E[e^{-\langle \phi_k, Z_{t_k}\rangle} | \mathcal{F}_{t_{k-1}}^Z]] \\
&\quad \geq E\Big[e^{-\sum_{i=1}^{k-1} \langle \phi_i, Z_{t_i}\rangle} \lim_{n\to\infty} E_{\phi_k}^{Y^n}[e^{-\langle Y_{t_k - t_{k-1}}^n, Z_{t_{k-1}}\rangle}]\Big] \qquad (6.17)\\
&\quad = \lim_{n\to\infty} E_{\phi_k}^{Y^n} \times E^Z[e^{-\sum_{i=1}^{k-2} \langle \phi_i, Z_{t_i}\rangle - \langle \phi_{k-1} + Y_{t_k - t_{k-1}}^n, Z_{t_{k-1}}\rangle}] \\
&\quad \geq \lim_{n\to\infty} E_{\phi_k}^{Y^n} \times E^X[e^{-\sum_{i=1}^{k-2} \langle \phi_i, X_{t_i}\rangle - \langle \phi_{k-1} + Y_{t_k - t_{k-1}}^n, X_{t_{k-1}}\rangle}],
\end{aligned}$$

where the inequality in (6.17) follows by Lemma 6.5, and the last inequality follows by the induction hypothesis. Now, for $\gamma \in (1/2, 1)$, we use conditioning and Proposition 2.3 in Mytnik (1998) to get

$$\begin{aligned}
&\lim_{n\to\infty} E_{\phi_k}^{Y^n} \times E^X[e^{-\sum_{i=1}^{k-2} \langle \phi_i, X_{t_i}\rangle - \langle \phi_{k-1} + Y_{t_k - t_{k-1}}^n, X_{t_{k-1}}\rangle}] \\
&\qquad = E\Big[e^{-\sum_{i=1}^{k-1} \langle \phi_i, X_{t_i}\rangle} \lim_{n\to\infty} E_{\phi_k}^{Y^n}[e^{-\langle Y_{t_k - t_{k-1}}^n, X_{t_{k-1}}\rangle}]\Big] \qquad (6.18)\\
&\qquad = E[e^{-\sum_{i=1}^k \langle \phi_i, X_{t_i}\rangle}],
\end{aligned}$$

and we are done for $\gamma \in (1/2, 1)$. For $\gamma = 1/2$, (6.18) follows immediately again by conditioning, and the fact that $Y = Y^n$ is a solution to the log-Laplace equation for super-Brownian motion. □

LEMMA 6.7. *For any nonnegative and Borel measurable function ψ on $\mathbb{R}_+ \times \mathbb{R}$*

$$E[e^{-\int_0^t \int_{\mathbb{R}} \psi(s,x) Z(s,x)\,dx\,ds}] \geq E[e^{-\int_0^t \int_{\mathbb{R}} \psi(s,x) X(s,x)\,dx\,ds}] \qquad \forall t \geq 0. \tag{6.19}$$

Before starting the proof, we recall the following definition.

DEFINITION 6.8. We say that a sequence $\psi_n(x)$ of functions converges bounded-pointwise to $\psi(x)$ provided $\lim_{n\to\infty} \psi_n(x) = \psi(x)$ for all x, and there exists a constant $K < \infty$ such that $\sup_{n,x} |\psi_n(x)| \leq K$.

PROOF OF LEMMA 6.7. First suppose that $\psi \in C_+(\mathbb{R}_+ \times \mathbb{R})$ is bounded. Then let us choose an approximating sequence of bounded functions $\phi_1^n, \dots, \phi_{k_n}^n \in C_+(\mathbb{R}_+)$ such that

$$\sum_{i=1}^{k_n} \langle \phi_i, f_{t_i} \rangle \to \int_0^t \int_{\mathbb{R}} \psi(s,x) f(s,x)\,ds\,dx \qquad \forall t \geq 0$$

for any $f \in D(\mathbb{R}_+, C_+(\mathbb{R}))$. In this way for bounded $\psi \in C_+(\mathbb{R}_+ \times \mathbb{R})$ the result follows immediately from Lemma 6.6. Now pass to the bounded-pointwise closure of this class of ψ's, that is the smallest class containing the above continuous ψ's which is closed under bounded-pointwise limits. Finally take monotone increasing limits to complete the proof. □

PROOF OF PROPOSITION 6.3. Take

$$\psi_n(s,x) = n 1_A(s,x).$$

Then by Lemma 6.7 we have

$$E[e^{-nZ(A)}] \geq E[e^{-nX(A)}],$$

where $Z(A) \equiv \int_A Z(s,x)\,dx\,ds$ and $X(A) \equiv \int_A X(s,x)\,dx\,ds$. Take $n \to \infty$ on both sides to get

$$P(Z(A) = 0) \geq P(X(A) = 0). \tag{6.20}$$

The required result follows immediately for A open because then

$$\{\mathrm{supp}(Z) \cap A = \varnothing\} = \{Z(A) = 0\}.$$

It then follows for compact A because

$$\{\mathrm{supp}(X) \cap A = \varnothing\} = \bigcup_n \{\mathrm{supp}(X) \cap A^{1/n} = \varnothing\},$$

where $A^{1/n}$ is the open set of points distance less than $1/n$ of A. The general result now follows by the inner regularity of the Choquet capacity $A \to P(\mathrm{supp}(Z) \cap A \neq \varnothing)$; see page 39 of Meyer (1966). $\square$

7. Bounds on the killing measure: Proof of Lemma 4.3. Let

$$G(\bar{U}^i) = \overline{\{(t,x) : \bar{U}^i(t,x) > 0\}}$$

be the closed graph of $\bar{U}^i$, and let

$$\Gamma_i^U(t) = \Gamma_i^U(t, \delta_0) = \{(s,x) : s_i \le s \le s_i + t, |x - x_i| \le (s - s_i)^{(1/2)-\delta_0} + \varepsilon^{1/2}\},$$

and let $\Gamma_j^V(t)$ be the corresponding set for V with (t_j, y_j) in place of (s_i, x_i). It is easy to check, using the definition of ρ_i, that

$$G(\bar{U}^i) \cap ([s_i, s_i + \rho_i] \times \mathbb{R}) \subset \Gamma_i^U(\rho_i). \tag{7.1}$$

Of course an analogous inclusion holds for $\bar{V}^j$. If $K'(\cdot)$ is a nondecreasing right-continuous $M_F(\mathbb{R})$-valued process, we let $S(K')$ denote the closed support of the associated random measure on space–time, $K'(ds, dx)$.

LEMMA 7.1. *$S(K^{i,U}) \subset G(\bar{U}^i)$ and $S(K^{j,V}) \subset G(\bar{V}^j)$ for all $i, j \in \mathbb{N}_\varepsilon$, P-a.s.*

PROOF. It is easy to see from (2.1) that $S(K^{i,U}) \subset [s_i, \infty) \times \mathbb{R}$. Let $\mathcal{O}$ be a bounded open rectangle in $((s_i, \infty) \times \mathbb{R}) \cap G(\bar{U}^i)^c$ whose corners have rational coordinates, and choose a smooth nonnegative function ϕ on $\mathbb{R}$ so that $\mathcal{O} = (r_1, r_2) \times \{\phi > 0\}$. Then $\bar{U}_r^i(\phi) = 0$ for all $r \in (r_1, r_2)$ and hence for all $r \in [r_1, r_2]$ a.s. by continuity. It then follows from (2.1) and $U^i \le \bar{U}^i$ that a.s.

$$0 = U_{r_2}^i(\phi) - U_{r_1}^i(\phi) = -(K_{r_2}^{i,U}(\phi) - K_{r_1}^{i,U}(\phi)).$$

Therefore $K^{i,U}(\mathcal{O}) = 0$. Taking unions over such open "rational" rectangles, we conclude that

$$K^{i,U}(G(\bar{U}^i)^c \cap ((s_i, \infty) \times \mathbb{R})) = 0 \qquad \text{a.s.}$$

On the other hand, from (2.6),

$$\begin{aligned} K^{i,U}(G(\bar{U}^i)^c \cap (\{s_i\} \times \mathbb{R})) &\le K^{i,U}(\{s_i\} \times [x_i - \sqrt{\varepsilon}, x_i + \sqrt{\varepsilon}]^c) \\ &= 0. \end{aligned}$$

In the last line we used (2.1) (recall from Section 2 this implies $U_s^i = 0$ for $s < s_i$) to see that $K_{s_i}^{i,U}(\cdot) \le \langle J^{x_i}, \cdot \rangle$. The last two displays imply that $K^{i,U}(G(\bar{U}^i)^c) = 0$ and hence the result for $K^{i,U}$. The proof for $K^{j,V}$ is the same. $\square$

Next we need a bound on the extinction times of nonnegative martingales which is a slight generalization of Lemma 3.4 of Mueller and Perkins (1992).

LEMMA 7.2. *Assume* $\gamma' = \gamma'' = \frac{1}{2}$ *or* $(\gamma', \gamma'') \in (1/2, 1) \times [1/2, 1]$. *Let* $M \geq 0$ *be a continuous* $(\mathcal{H}_t)$*-local martingale and* T *be an* $(\mathcal{H}_t)$*-stopping time so that for some* $\delta \geq 0$ *and* $c_0 > 0$,

$$\frac{d\langle M\rangle_t}{dt} \geq c_0 \mathbf{1}(t < T) M_t^{2\gamma'} (t+\delta)^{(1/2)-\gamma''} \qquad \text{for } t > 0. \tag{7.2}$$

If $\tau_M(0) = \inf\{t \geq 0 : M_t = 0\}$, *then there is a* $c_{7.2}(\gamma') > 0$ *such that*

$$P(T \wedge \tau_M(0) \geq t | \mathcal{H}_0) \leq c_{7.2}(\gamma') c_0^{-1} M_0^{2-2\gamma'} t^{\gamma'' - (3/2)} \qquad \text{for all } t \geq \delta/2.$$

PROOF. If $\gamma' = \gamma'' = \frac{1}{2}$, the lemma follows from a slight extension of the proof of Lemma 5.1, so assume $\gamma' \in (1/2, 1)$. Let $V = T \wedge \tau_M(0)$. As usual there is a Brownian motion $B(t)$ such that $M(t) = B(\langle M\rangle_t)$ for $t \leq V$. By (7.2) we have

$$\begin{aligned} \int_0^V c_0 (t+\delta)^{(1/2)-\gamma''}\, dt &\leq \int_0^V M_t^{-2\gamma'}\, d\langle M\rangle_t \\ &\leq \int_0^{\langle M\rangle_V} B_u^{-2\gamma'}\, du \leq \int_0^{\tau_B(0)} B_u^{-2\gamma'}\, du. \end{aligned}$$

If $L_t^x, x \in \mathbb{R}, t \geq 0$ is the semimartingale local time of B, the Ray–Knight theorem [see Theorem VI.52.1 in Rogers and Williams (1987)] and the occupation time formula implies that the above gives

$$\begin{aligned} &E[(V+\delta)^{(3/2)-\gamma''} - \delta^{(3/2)-\gamma''} | \mathcal{H}_0] \\ &\quad \leq ((3/2) - \gamma'') c_0^{-1} \int_0^\infty x^{-2\gamma'} E(L^x_{\tau_B(0)} | B_0)\, dx \\ &\quad = ((3/2) - \gamma'') c_0^{-1} \int_0^\infty x^{-2\gamma'} 2(M_0 \wedge x)\, dx \\ &\quad \leq c_1(\gamma') c_0^{-1} M_0^{2-2\gamma'} \qquad (\text{use } \gamma' > 1/2). \end{aligned} \tag{7.3}$$

A bit of calculus shows that

$$(t+\delta)^{(3/2)-\gamma''} - \delta^{(3/2)-\gamma''} \geq \tfrac{1}{2}(\sqrt{3} - \sqrt{2}) t^{(3/2)-\gamma''} \qquad \text{for all } t \geq \delta/2. \tag{7.4}$$

Therefore by (7.3) and (7.4), for $t \geq \delta/2$,

$$\begin{aligned} P(V \geq t | \mathcal{H}_0) &\leq \frac{E[(V+\delta)^{(3/2)-\gamma''} - \delta^{(3/2)-\gamma''} | \mathcal{H}_0]}{(t+\delta)^{(3/2)-\gamma''} - \delta^{(3/2)-\gamma''}} \\ &\leq \frac{2c_1(\gamma') c_0^{-1} M_0^{2-2\gamma'}}{(\sqrt{3} - \sqrt{2}) t^{(3/2)-\gamma''}} \\ &\equiv c_{7.2} c_0^{-1} M_0^{2-2\gamma'} t^{\gamma''-(3/2)}. \end{aligned}$$

□

Define $\rho_j^V = \rho_j^{V,\delta_0,\varepsilon}$ just as ρ_i but with $\bar V^j_{t_j+t}$ in place of $\bar U^i_{s_i+t}$ and y_j in place of x_i.

LEMMA 7.3. *$Q_i(\bigcup_{j=1}^{pN_\varepsilon}\{\rho_j^V \le t\}) \le c_{4.4}(t\vee\varepsilon)p\mathbf{1}(p\ge\varepsilon)$ for all $\varepsilon, p, t \in (0,1]$ and $i \in \mathbb{N}_\varepsilon$.*

PROOF. All the P-local martingales and P-white noises arising in the definition of $\{\bar V^j, j\in\mathbb{N}_\varepsilon\}$ remain such under Q_i because they are all orthogonal to

$$\frac{dQ_i}{dP}\Big|_{\mathcal{F}_t} = \mathbf{1}(t<s_i) + \mathbf{1}(t\ge s_i)\frac{\bar U^i_{t\wedge(s_i+\bar\tau_i)}(1)}{\varepsilon}.$$

The proof of Lemma 4.4 for $\{\rho_i\}$ under P therefore applies to $\{\rho_j^V\}$ under Q_i. □

Recall we are trying to show that the killing measure $K_t^{i,U}$ associated with the i cluster of U grows slowly enough for small t. We will control the amount of killing here by controlling the amount of killing by the V^j's. The following result essentially shows that with high probability for small t, there is no killing during $[s_i, s_i+t]$ from the V^j's which are born before time s_i. Note it is particularly important that there is no V mass on the birth site of the U^i cluster.

Recall from (4.1) that $\bar\delta = \bar\delta(\gamma) = \frac{1}{3}(\frac{3}{2} - 2\gamma)$. We introduce

$$\underline{\rho}_i^V = \min_{j:t_j\le s_i} \rho_j^V.$$

LEMMA 7.4. *There is a constant $c_{7.4}(\gamma) > 0$ so that for $0<\delta_0\le\bar\delta(\gamma)$,*

$$Q_i\left(\Gamma_i^U(t)\cap\left\{\bigcup_{j:t_j\le s_i} G(\bar V^j)\right\}\ne\varnothing, \underline{\rho}_i^V > 2t\right) \le c_{7.4}(\gamma)(\varepsilon\vee t)^{\bar\delta}$$

for all $\varepsilon, t\in(0,1]$ and $s_i\le t$.

PROOF. Assume ε, t, s_i and δ_0 are as above. Set $\alpha = \frac{1}{2} - \delta_0 (\ge \frac{1}{3})$ and choose $n_0 \le n_1 \in \mathbb{Z}_+$ so that

$$2^{-n_0-1} < t\vee\varepsilon \le 2^{-n_0}, \qquad 2^{-n_1-1} < \varepsilon \le 2^{-n_1}. \tag{7.5}$$

Assume that

$$\underline{\rho}_i^V > 2t, \tag{7.6}$$

until otherwise indicated. Suppose $t_j \le s_i$ (hence $t_j < s_i$) and

$$(t_j, y_j) \notin [0, s_i) \times [x_i - 7\cdot 2^{-n_0\alpha}, x_i + 7\cdot 2^{-n_0\alpha}].$$

Then

$$|y_j - x_i| > 7 \cdot 2^{-n_0\alpha} \geq 7(t \vee \varepsilon)^\alpha \geq t^\alpha + (t + s_i - t_j)^\alpha + 2\sqrt{\varepsilon},$$

and so

$$\Gamma_i^U(t) \cap \Gamma_j^V(s_i + t - t_j) = \varnothing.$$

By (7.6) we have $\rho_j^V > s_i - t_j + t$, and so by (7.1), or more precisely its analogue for $\bar{V}^j$, we have

$$\Gamma_i^U(t) \cap G(\bar{V}^j) \subset \Gamma_i^U(t) \cap \Gamma_j^V(s_i + t - t_j) = \varnothing. \tag{7.7}$$

We therefore have shown that, assuming (7.6),

$$\begin{aligned} &\{(t_j, y_j) : t_j \leq s_i, \Gamma_i^U(t) \cap G(\bar{V}^j) \neq \varnothing\} \\ &\qquad \subset [0, s_i) \times [x_i - 7 \cdot 2^{-n_0\alpha}, x_i + 7 \cdot 2^{-n_0\alpha}]. \end{aligned} \tag{7.8}$$

Next we cover the rectangle on the right-hand side of the above by rectangles as follows:

$$\begin{aligned} R_n^0 &= [s_i - 2^{-n+1}, s_i - 2^{-n}] \times [x_i - 7 \cdot 2^{-n\alpha}, x_i + 7 \cdot 2^{-n\alpha}], \\ R_n^r &= [s_i - 2^{-n}, s_i] \times [x_i + 7 \cdot 2^{-(n+1)\alpha}, x_i + 7 \cdot 2^{-n\alpha}], \\ R_n^\ell &= [s_i - 2^{-n}, s_i] \times [x_i - 7 \cdot 2^{-n\alpha}, x_i - 7 \cdot 2^{-(n+1)\alpha}]. \end{aligned}$$

Then it is easy to check that

$$\begin{aligned} &\bigcup_{n=n_0}^{\infty} (R_n^0 \cup R_n^r \cup R_n^\ell) \\ &\qquad \supset [s_i - 2^{-n_0+1}, s_i) \times [x_i - 7 \cdot 2^{-n_0\alpha}, x_i + 7 \cdot 2^{-n_0\alpha}] \end{aligned} \tag{7.9}$$

$$\supset [0, s_i) \times [x_i - 7 \cdot 2^{-n_0\alpha}, x_i + 7 \cdot 2^{n_0\alpha}]. \tag{7.10}$$

We group together those $\bar{V}^j$'s which have their initial "seeds" in each of the above rectangles. That is, for $q = 0, \ell, r$ consider

$$\begin{aligned} V^{n,q}(t,x) &= \sum_j \mathbf{1}((t_j, y_j) \in R_n^q) V^j(t,x), \\ \widetilde{V}^{n,q}(t,x) &= \sum_j \mathbf{1}((t_j, y_j) \in R_n^q) \widetilde{V}^j(t,x), \\ \bar{V}^{n,q}(t,x) &= \sum_j \mathbf{1}((t_j, y_j) \in R_n^q) \bar{V}^j(t,x). \end{aligned}$$

We also let $V_t^{n,q}$, $\widetilde{V}_t^{n,q}$ and $\bar{V}_t^{n,q}$ denote the corresponding measure-valued processes.

It follows from (7.8) and (7.10) that

$$\begin{aligned}Q_i&\Biggl(\bigcup_{t_j\le s_i}(G(\bar V^j)\cap\Gamma_i^U(t)\ne\varnothing,\underline{\rho}_i^V>2t)\Biggr)\\ &\le\sum_{n=n_0}^{n_1}\sum_{q=0,r,\ell}Q_i(G(\bar V^{n,q})\cap\Gamma_i^U(t)\ne\varnothing,\underline{\rho}_i^V>2t)\\ &\quad+Q_i\Biggl(\bigcup_{n=n_1+1}^{\infty}\bigcup_{q=0,r,l}(G(\bar V^{n,q})\cap\Gamma_i^U(t))\ne\varnothing\Biggr).\end{aligned}\tag{7.11}$$

We will use different arguments to show that each of the two terms on the right-hand side of (7.11) is small. For the second term a very crude argument works. Namely, for the supports of the $\bar V^j$ clusters with initial "seeds" in $\bigcup_{n=n_1+1}^{\infty}(R_n^0\cup R_n^r\cup R_n^\ell)$ to intersect the support of U^i, the $\bar V^j$ clusters must be born in $\bigcup_{n=n_1+1}^{\infty}(R_n^0\cup R_n^r\cup R_n^\ell)$, and the probability of this event is already small. More precisely,

$$\begin{aligned}Q_i&\Biggl(\bigcup_{n=n_1+1}^{\infty}\bigcup_{q=0,r,l}(G(\bar V^{n,q})\cap\Gamma_i^U(t))\ne\varnothing\Biggr)\\ &\le Q_i\Biggl(\eta_\varepsilon^-\Biggl(\bigcup_{n=n_1+1}^{\infty}(R_n^0\cup R_n^r\cup R_n^\ell)\Biggr)>0\Biggr).\end{aligned}\tag{7.12}$$

By Proposition 5.2 and the decomposition for $\bar U^i(1)$ in (2.6) [see also (5.1)], we have

$$\begin{aligned}&Q_i((x_i,y_j)\in A)\\ &\quad=E_P\biggl(\frac{\bar U^i_{s_i+[(t_j-s_i)^+\wedge\bar\tau_i]}(1)}{\varepsilon}\mathbf{1}((x_i,y_j)\in A)\biggr)=P((x_i,y_j)\in A).\end{aligned}\tag{7.13}$$

This and the analogue of (7.9) with n_1+1 in place of n_0, implies that the right-hand side of (7.12) is at most

$$\begin{aligned}&Q_i(\eta_\varepsilon^-([s_i-2^{-n_1},s_i)\times[x_i-7\cdot2^{-(n_1+1)\alpha},x_i+7\cdot2^{-(n_1+1)\alpha}])>0)\\ &\quad\le2(14\cdot2^{-(n_1+1)\alpha})\le42\varepsilon^\alpha.\end{aligned}\tag{7.14}$$

Substitute this bound into (7.11) to get

$$\begin{aligned}Q_i&\Biggl(\bigcup_{t_j\le s_i}(G(\bar V^j)\cup\Gamma_i^U(t))\ne\varnothing,\underline{\rho}_i^V>2t\Biggr)\\ &\le\sum_{n=n_0}^{n_1}\sum_{q=0,r,\ell}Q_i(G(\bar V^{n,q})\cap\Gamma_i^U(t)\ne\varnothing,\underline{\rho}_i^V>2t)+42\varepsilon^\alpha.\end{aligned}\tag{7.15}$$

Now we are going to bound each term in the sum on the right-hand side of (7.15). To this end, in what follows, we assume that $n_0 \le n \le n_1$, and, for $q = 0, r, l$, set

$$N_t^{n,q} = \sum_j \mathbf{1}((t_j, y_j) \in R_n^q) \times \int_0^t \int_{\mathbb{R}} \biggl(V(s,x)^{2\gamma-1} V^j(s,x) + (\bar{V}(s,x)^{2\gamma} - V(s,x)^{2\gamma}) \frac{\widetilde{V}^j(s,x)}{\widetilde{V}(s,x)} \biggr)^{1/2} \bar{W}^{j,V}(ds,dx). \tag{7.16}$$

Note that $N^{n,q}$ is a continuous local martingale under Q_i.

The treatment of the cases $q = 0$ and $q = r, l$ is different. First, let $q = 0$. Basically, in this case, we will show that, the on the event $\{\underline{\rho}_i^V > 2t\}$, the total mass of $\bar{V}^{n,0}$ dies out with high probability before the time s_i (and, in fact, even before $s_i - 2^{-n-1}$). Hence, with this high probability, the support of $\bar{V}^{n,0}$ does not intersect Γ_i^U. Let us make this precise. We have from (2.6)

$$\bar{V}^{n,0}_{t+(s_i-2^{-n})^+}(1) = \bar{V}^{n,0}_{(s_i-2^{-n})^+}(1) + \bar{M}_t^{n,0}, \tag{7.17}$$

where

$$\bar{V}^{n,0}_{(s_i-2^{-n})^+}(1) = \int\int \mathbf{1}((s,y) \in R_n^0) \eta_\varepsilon^-(ds,dy) + N^{n,0}_{(s_i-2^{-n})^+}$$

and

$$\bar{M}_t^{n,0} = N^{n,0}_{t+(s_i-2^{-n})^+} - N^{n,0}_{(s_i-2^{-n})^+} \tag{7.18}$$

is a continuous $\mathcal{F}_{t+(s_i-2^{-n})^+}$-local martingale under Q_i.

Assume for now that $s_i > 2^{-n}$ since otherwise $\bar{V}^{n,0}_{s_i}(1) = 0$ and the bound (7.22) below is trivial. An easy localization argument shows that (recall that $n_0 \le n \le n_1$)

$$\begin{aligned} &Q_i(\bar{V}^{n,0}_{(s_i-2^{-n})}(1) \ge 2^{-n(1+\alpha-\bar{\delta})}) \\ &\quad \le 2^{n(1+\alpha-\bar{\delta})} Q_i\biggl(\int\int \mathbf{1}((s,y) \in R_n^0)\eta_\varepsilon^-(ds,dy)\biggr) \\ &\quad \le 2^{n(1+\alpha-\bar{\delta})} \varepsilon[\varepsilon^{-1}2^{-n} + 1] 14 \cdot 2^{-n\alpha} \qquad \text{[by (7.13)]} \\ &\quad \le 14(2^{-n\bar{\delta}})(2^n\varepsilon + 1) \le 28 \cdot 2^{-n\bar{\delta}}. \end{aligned} \tag{7.19}$$

Now from (7.16) and (7.18), if $t' \equiv s_i - 2^{-n} + t < T' \equiv \min_{j:t_j \le s_i}(\rho_j^V + t_j)$, then

$$
\begin{aligned}
&\frac{d}{dt}\langle \bar{M}^{n,0}\rangle_t \\
&\quad = \int V(t',x)^{2\gamma-1}\bar{V}^{n,0}(t',x) + (\bar{V}(t',x)^{2\gamma} - V(t',x)^{2\gamma})\frac{\widetilde{V}^{n,0}(t',x)}{\widetilde{V}(t',x)}\,dx \\
&\quad \ge \int V^{n,0}(t',x)^{2\gamma} + \widetilde{V}^{n,0}(t',x)^{2\gamma}\,dx \qquad (7.20)\\
&\quad \ge 2^{-2\gamma}\int \bar{V}^{n,0}(t',x)^{2\gamma}\mathbf{1}(|x-x_i| \le 7\cdot 2^{-n\alpha} + (2^{-n}+t)^{\alpha} + \sqrt{\varepsilon})\,dx \\
&\quad \ge 2^{-2\gamma}\bar{V}_{t'}^{n,0}(1)^{2\gamma}(2[7\cdot 2^{-n\alpha} + (2^{-n}+t)^{\alpha} + \sqrt{\varepsilon}])^{1-2\gamma}.
\end{aligned}
$$

In the last line we used Jensen's inequality and the fact that $T' > t'$ implies $\bar{V}^{n,0}(t',\cdot)$ is supported in the closed interval with endpoints $x_i \pm (7\cdot 2^{-n\alpha} + (t+2^{-n})^{\alpha} + \sqrt{\varepsilon})$. A bit of arithmetic (recall $2^{-n} \ge \varepsilon$ for $n \le n_1$) shows that (7.20) implies for some $c(\gamma) > 0$,

$$
\begin{aligned}
&\frac{d}{dt}\langle \bar{M}^{n,0}\rangle_t \ge c(\gamma)(\bar{V}_{t+(s_i-2^{-n})}^{n,0}(1))^{2\gamma}[2^{-n}+t]^{\alpha(1-2\gamma)} \\
&\qquad\qquad \text{for } t < T \equiv \Big(\min_{j:t_j\le s_i}(\rho_j^V + t_j) - (s_i - 2^{-n})\Big)^{+}.
\end{aligned} \tag{7.21}
$$

Note that T is an $\mathcal{F}_{(s_i-2^{-n})+t}$-stopping time. Therefore (7.21) allows us to apply Lemma 7.2 to $t \to \bar{V}_{(s_i-2^{-n})+t}^{n,0}(1) \equiv M_t$ with $\gamma' = \gamma$, $\gamma'' = \gamma - \delta_0(2\gamma-1)$ and $\delta = 2^{-n}$. Here notice that $\delta_0 \le 1/6$ implies $\gamma'' \in [\frac{1}{2}, \frac{3}{4}]$ and $\gamma'' = 1/2$ if $\gamma = 1/2$. Therefore, Lemma 7.2, the fact that $\underline{\rho}_i^V > 2t$ implies $T > t \ge s_i > 2^{-n}$, and (7.19) imply

$$
\begin{aligned}
&Q_i(\bar{V}_{s_i-2^{-n-1}}^{n,0}(1) > 0, \underline{\rho}_i^V > 2t) \\
&\quad \le Q_i(\bar{V}_{s_i-2^{-n}}^{n,0}(1) \ge 2^{-n(1+\alpha-\bar{\delta})}) \\
&\qquad + E_{Q_i}[Q_i(T\wedge\tau_M(0) \ge 2^{-n-1}|\mathcal{F}_{s_i-2^{-n}})\mathbf{1}(\bar{V}_{s_i-2^{-n}}^{n,0}(1) < 2^{-n(1+\alpha-\bar{\delta})})] \\
&\quad \le 28\cdot 2^{-n\bar{\delta}} + c_{7.2}(\gamma)c(\gamma)^{-1}2^{-n(1+\alpha-\bar{\delta})(2-2\gamma)}2^{-(n+1)(\gamma-\delta_0(2\gamma-1)-(3/2))} \qquad (7.22)\\
&\quad \le c'(\gamma)(2^{-n\bar{\delta}} + 2^{-n((3/2)-2\gamma-2(1-\gamma)\bar{\delta}-\delta_0)}) \qquad \text{(by the definition of } \alpha) \\
&\quad \le c'(\gamma)(2^{-n\bar{\delta}} + 2^{-n(3\bar{\delta}-2(1-\gamma)\bar{\delta}-\delta_0)}) \qquad \text{(by the definition of } \bar{\delta}) \\
&\quad \le c_0(\gamma)2^{-n\bar{\delta}},
\end{aligned}
$$

where $\delta_0 \le \bar{\delta}$ and $\gamma \ge 1/2$ are used in the last line.

Next consider $\bar{V}^{n,r}$. The analogue of (7.17) now is

$$\bar{V}^{n,r}_{s_i+t}(1) = \bar{V}^{n,r}_{s_i}(1) + \bar{M}^{n,r}_t,$$

where

$$\bar{M}^{n,r}_t = N^{n,r}_{s_i+t} - N^{n,r}_{s_i}.$$

An argument similar to the derivation of (7.19) shows that

$$Q_i(\bar{V}^{n,r}_{s_i}(1) \geq 2^{-n(1+\alpha-\bar{\delta})}) \leq 28 \cdot 2^{-n\bar{\delta}}. \tag{7.23}$$

Next we argue as in (7.20) and (7.21) to see that for $s_i + t < T' \equiv \min_{j:t_j \leq s_i}(\rho^V_j + t_j)$,

$$\begin{aligned}\frac{d}{dt}\langle \bar{M}^{n,r}\rangle_t &\geq 2^{-2\gamma}\bar{V}^{n,r}_{s_i+t}(1)^{2\gamma}([7\cdot 2^{-(n+1)\alpha} + (2^{-n}+t)^{(1/2)-\delta_0} + \sqrt{\varepsilon}]2)^{1-2\gamma}\\ &\geq c'(\gamma)(\bar{V}^{n,r}_{s_i+t}(1))^{2\gamma}(2^{-n}+t)^{\alpha(1-2\gamma)},\end{aligned}$$

where we again used $n_0 \leq n \leq n_1$. Now we apply Lemma 7.2 and (7.23), as in the derivation of (7.22), to conclude that

$$Q_i(\bar{V}^{n,r}_{s_i+2^{-n}}(1) > 0, \underline{\rho}^V_i > 2t) \leq c_1(\gamma)2^{-n\bar{\delta}}. \tag{7.24}$$

If $\bar{V}^{n,r}_{s_i+2^{-n}}(1) = 0$, then $\bar{V}^{n,r}_u(1) = 0$ for all $u \geq s_i + 2^{-n}$, and so if in addition, $\underline{\rho}^V_i > 2t$, then by the definition of ρ^V_j,

$$\begin{aligned} G(\bar{V}^{n,r}) \subset \{(s,x)\colon & s_i - 2^{-n} \leq s \leq s_i + 2^{-n},\\ & 7\cdot 2^{-(n+1)\alpha} - (s - s_i + 2^{-n})^\alpha - \sqrt{\varepsilon}\\ & \leq x - x_i \leq 7 \cdot 2^{-n\alpha} + (s - s_i + 2^{-n})^\alpha + \sqrt{\varepsilon}\}.\end{aligned} \tag{7.25}$$

A bit of algebra (using our choice of the factor 7 and $n_0 \leq n \leq n_1$) shows that

$$x_i + 2^{-n\alpha} + \sqrt{\varepsilon} < x_i + 7\cdot 2^{-(n+1)\alpha} - (2^{-n} + 2^{-n})^\alpha - \sqrt{\varepsilon},$$

and so the set on the right-hand side of (7.25) is disjoint from $\Gamma^U_i(t)$. Therefore by (7.24) we may conclude that

$$Q_i(G(\bar{V}^{n,r}) \cap \Gamma^U_i(t) \neq \varnothing, \underline{\rho}^V_i > 2t) \leq c_1(\gamma)2^{-n\bar{\delta}}. \tag{7.26}$$

Of course the same bound holds for $G(\bar{V}^{n,\ell})$.

Note that $\bar{V}^{n,0}_{s_i-2^{-n-1}}(1) = 0$ implies $\bar{V}^{n,0}_s(1) = 0$ for all $s \geq s_i - 2^{-n-1}$ and so $G(\bar{V}^{n,0}) \cap \Gamma^U_i(t)$ is empty. Therefore (7.22) and (7.26) show that the summation on the right-hand side of (7.15) is at most

$$\sum_{n=n_0}^{n_1} (c_0(\gamma)+2c_1(\gamma))2^{-n\bar\delta} \le c_2(\gamma)(t\vee\varepsilon)^{\bar\delta}.$$

We substitute the above into (7.15) to see that

$$\begin{aligned} &Q_i\biggl(\bigcup_{t_j\le s_i}(G(\bar V^j)\cup\Gamma_i^U(t))\neq\varnothing,\underline{\rho}_i^V>2t\biggr)\\ &\qquad\le 42\varepsilon^\alpha + c_2(\gamma)(t\vee\varepsilon)^{\bar\delta}\le c_{7.4}(\gamma)(t\vee\varepsilon)^{\bar\delta}.\end{aligned}$$

In the last line we used $\bar\delta\le 1/6<1/4\le\alpha$. □

Proof of Lemma 4.3. Fix $0<\delta_0\le\bar\delta$, $t\in(0,1]$ and assume $s_i,s\le t$. By (7.1) and Lemma 7.1 on $\{\rho_i>s\}$ we have

$$K^{i,U}_{s_i+s}(1)=K^{i,U}(\Gamma_i^U(s))\le\sum_j K^{j,V}(\Gamma_i^U(s)),$$

where (2.2) is used in the last inequality. Next use $S(K^{j,V})\subset G(\bar V^j)$ (by Lemma 7.1) and $S(K^{j,V})\subset[t_j,\infty)\times\mathbb{R}$ to conclude that on

$$\{\rho_i>s\}\cap\biggl\{\biggl(\bigcup_{t_j\le s_i}G(\bar V^j)\biggr)\cap\Gamma_i^U(t)=\varnothing\biggr\}\equiv\{\rho_i>s\}\cap D_i(t),$$

we have

$$K^{i,U}_{s_i+s}(1)\le\sum_j \mathbf{1}(s_i<t_j\le s_i+s)K^{j,V}(\Gamma_i^U(s)). \tag{7.27}$$

Another application of (7.1) and Lemma 7.1, this time to $\bar V^j$, shows that for $t_j>s_i$,

$$S(K^{j,V})\cap([0,s_i+s]\times\mathbb{R})\subset\Gamma_j^V(s_i+s-t_j)\qquad\text{on }\{\rho_j^V>s\}. \tag{7.28}$$

An elementary calculation shows that

$$\begin{aligned}&\Gamma_i^U(s)\cap\Gamma_j^V(s_i+s-t_j)=\varnothing\\ &\quad\text{for } s_i<t_j\le s_i+s \text{ and } |y_j-x_i|>2(\sqrt{\varepsilon}+s^{(1/2)-\delta_0}).\end{aligned} \tag{7.29}$$

If $F_i(t)=\bigcap_{j:t_j\le s_i+t}\{\rho_j^V>2t\}$, then use (7.28) and (7.29) in (7.27) to see that on $D_i(t)\cap F_i(t)$, for $s<t\wedge\rho_i$,

$$\begin{aligned}&K^{i,U}_{s_i+s}(1)\\ &\quad\le\sum_j\mathbf{1}(s_i<t_j\le s_i+s,|y_j-x_i|\le 2(\sqrt{\varepsilon}+s^{(1/2)-\delta_0}))K^{j,V}_{s_i+s}(1)\\ &\quad\equiv L^i(s).\end{aligned} \tag{7.30}$$

Note that L^i is a nondecreasing process. If we sum the second equation in (2.1) over j satisfying $s_i < t_j \le s_i + s$, $|y_j - x_i| \le 2(\sqrt{\varepsilon} + s^{(1/2)\delta_0})$, and denote this summation by $\sum_j^{(i)}$, then

$$\begin{aligned} L^i(s) &\le \sum_j^{(i)} K^{j,V}_{s_i+s}(1) + V^j_{s_i+s}(1) \\ &= \int\int \mathbf{1}(s_i < t' \le s_i + s, |y' - x_i| \le 2(\sqrt{\varepsilon} + s^{(1/2)-\delta_0}))\eta^-_\varepsilon(dt', dy') \\ &\quad + \sum_j^{(i)} \int_0^{s_i+s} \int_{\mathbb{R}} V(s', x)^{\gamma-(1/2)} V^j(s', x)^{1/2} W^{j,V}(ds', dx). \end{aligned} \tag{7.31}$$

Now take means in (7.31), use (7.13) and use a standard localization argument to handle the Q_i martingale term, to conclude that

$$\begin{aligned} &E_{Q_i}(L^i(s)) \\ &\quad \le E_{Q_i}\left(\int\int \mathbf{1}(s_i < t' \le s_i + s, |y' - x_i| \le 2(\sqrt{\varepsilon} + s^{(1/2)-\delta_0}))\eta^-_\varepsilon(dt', dy')\right) \\ &\quad = \sum_j \mathbf{1}(s_i < j\varepsilon \le s_i + s)\varepsilon \\ &\qquad\quad \times \int_0^1 \int_0^1 \int_{y_j-\sqrt{\varepsilon}}^{y_j+\sqrt{\varepsilon}} J((y_j - y')\varepsilon^{-1/2})\varepsilon^{-1/2} \\ &\qquad\qquad\qquad \times \mathbf{1}(|y' - x_i| \le (2\sqrt{\varepsilon} + 2s^{(1/2)-\delta_0}))\, dy'\, dy_j\, dx_i \\ &\quad \le \sum_j \mathbf{1}(s_i < j\varepsilon \le s_i + s)\varepsilon \int_0^1 \int_0^1 \mathbf{1}(|y_j - x_i| \le (3\sqrt{\varepsilon} + 2s^{(1/2)-\delta_0}))\, dy_j\, dx_i \\ &\quad \le 2(3\sqrt{\varepsilon} + 2s^{(1/2)-\delta_0})\left(\sum_j \mathbf{1}(s_i < j\varepsilon \le s_i + s)\varepsilon\right) \\ &\quad \le 6(\sqrt{\varepsilon} + s^{(1/2)-\delta_0})(s + \varepsilon) \le 12(s + \varepsilon)^{(3/2)-\delta_0}. \end{aligned}$$

We take $s = 2^{-n}$ in the above, use Markov's inequality, and sum over n to conclude that for some $c(\delta_0) > 0$ independent of ε,

$$Q_i\left(L^i(2^{-n}) \le \left(\frac{2^{-n-1} + \varepsilon}{2}\right)^{(3/2)-2\delta_0} \text{ for } N \le n \le \log_2(1/\varepsilon)\right) \ge 1 - c(\delta_0)2^{-N\delta_0}.$$

Recall that $L^i(\cdot)$ is nondecreasing and consider $s \in [2^{-n-1}, 2^{-n}]$ to see that above implies that for $2^{-N} \geq \varepsilon$,

$$Q_i(L^i(s) \leq (s+\varepsilon)^{(3/2)-2\delta_0} \text{ for all } s \in [0, 2^{-N}]) \geq 1 - c(\delta_0) 2^{-N\delta_0}.$$

An easy interpolation argument in N now shows that for some $c_0(\delta_0)$, independent of ε,

$$Q_i(L^i(s) \leq (s+\varepsilon)^{(3/2)-2\delta_0} \text{ for } 0 \leq s \leq u) \geq 1 - c_0(\delta_0)(u \vee \varepsilon)^{\delta_0} \qquad \forall u \geq 0. \tag{7.32}$$

Apply (7.32) in (7.30) and conclude

$$\begin{aligned} Q_i(\theta_i < \rho_i \wedge t) &\leq Q_i(K^{i,U}_{s_i+s}(1) > (s+\varepsilon)^{(3/2)-2\delta_0} \ \exists s < \rho_i \wedge t) \\ &\leq Q_i(F_i(t)^c) + Q_i(D_i(t)^c \cap F_i(t)) \\ &\quad + Q_i(L^i(s) > (s+\varepsilon) \ \exists s < \rho_i \wedge t) \\ &\leq Q_i\Bigg(\bigcup_{j \leq (2t/\varepsilon) \wedge N_\varepsilon} \{\rho_j^V \leq 2t\}\Bigg) + Q_i(D_i(t)^c \cap \{\underline{\rho}_i^V > 2t\}) \\ &\quad + c_0(\delta_0)(t \vee \varepsilon)^{\delta_0}. \end{aligned} \tag{7.33}$$

Recall from Section 1 that $N_\varepsilon = \lfloor \varepsilon^{-1} \rfloor$. The second term is at most $c_{7.4}(\varepsilon \vee t)^{\bar{\delta}}$ by Lemma 7.4, and by Lemma 7.3, if $4t \leq 1$ and $\varepsilon \leq 1/2$, the first term is at most

$$Q_i\Bigg(\bigcup_{j \leq 4tN_\varepsilon} \{\rho_j^V \leq 2t\}\Bigg) \leq 8c_{4.4}(t \vee \varepsilon)t \leq 8c_{4.4}(t \vee \varepsilon).$$

If $4t > 1$ or $\varepsilon > 1/2$, the above bound is trivial as $c_{4.4} \geq 1$. We conclude from (7.33) that

$$Q_i(\theta_i < \rho_i \wedge t) \leq 8c_{4.4}(t \vee \varepsilon) + c_{7.4}(\varepsilon \vee t)^{\bar{\delta}} + c_0(\delta_0)(t \vee \varepsilon)^{\delta_0}.$$

The result follows because $\delta_0 \leq \bar{\delta} \leq 1$. $\square$

APPENDIX A: MOMENT BOUNDS, TIGHTNESS AND PROOF OF PROPOSITION 2.2

We start with a moment bound obtained by a modification of the proof of Lemma 4.2 in Mueller and Perkins (1992). Let $p(t,x) = p_t(x)$ denote that Gaussian kernel, that is,

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}, \qquad t > 0, x \in \mathbb{R}. \tag{A.1}$$

Let S_t denote the corresponding semigroup, so $S_t f = p_t * f$ for appropriate functions f.

LEMMA A.1. *For any $q \geq 1$ and $\lambda, T > 0$ there is a $C_{T,\lambda,q}$ such that for all $\varepsilon \in (0,1]$:*

(a) $\sup_{t \leq T} \int e^{\lambda|x|} E(\bar{U}(t,x)^q + \bar{V}(t,x)^q)\, dx \leq C_{T,\lambda,q}$,
(b) $\sup_{t \leq T, x \in \mathbb{R}} e^{\lambda|x|} E(\bar{U}(t,x)^q + \bar{V}(t,x)^q) \leq C_{T,\lambda,q}$.

REMARK A.2. Lemma A.1 and Theorem 1.1 of Mytnik (1998) easily imply uniqueness in law of each of $\bar{U}$ and $\bar{V}$ separately for a pair $(\bar{U}, \bar{V})$ solving (2.7). To show the uniqueness in law for the pair $(\bar{U}, \bar{V})$, one should follow the proof of Theorem 1.1 of Mytnik (1998) and derive the counterpart of Proposition 2.3 from Mytnik (1998), which is the main ingredient of the proof. More specifically, suppose $t \in [s_i, t_i)$ for some $i \in \mathbb{N}_\varepsilon$. Following the argument from Mytnik (1998), for any nonnegative $\phi_1, \phi_2 \in L^1(\mathbb{R})$, one can easily construct a sequence of $M_F(\mathbb{R})^2$-valued processes $\{(Y^{1,n}, Y^{2,n})\}_{n \geq 0}$ such that $\{Y^{1,n}\}_{n \geq 1}$ and $\{Y^{2,n}\}_{n \geq 1}$ are independent, and for any $(\bar{U}, \bar{V})$ solving (2.7) we have

$$\begin{aligned} &E[e^{-\langle \phi_1, \bar{U}_t\rangle + \langle \phi_2, \bar{V}_t\rangle}] \\ &\quad = \lim_{n\to\infty} E[e^{-\langle Y^{1,n}_{t-s_i}, \bar{U}_{s_i}\rangle + \langle Y^{2,n}_{t-s_i}, \bar{V}_{s_i}\rangle} | Y^{1,n}_0 = \phi_1, Y^{2,n}_0 = \phi_2]. \end{aligned} \tag{A.2}$$

A similar expression can be derived for $t \in [t_i, s_{i+1}), i \in \mathbb{N}_\varepsilon$, and then uniqueness in law for the pair $(\bar{U}, \bar{V})$ follows by standard argument: see again Mytnik (1998) where the single process without immigration is treated.

PROOF OF LEMMA A.1. It suffices to consider $\bar{U}$. We let C denote a constant which may depend on q, λ and T, and which may change from line to line. Note that equation (2.7) for $\bar{U}$ can be rewritten in the so-called mild form [see Theorem 2.1 of Shiga (1994)]

$$\begin{aligned} \bar{U}_t(x) &= \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\eta_\varepsilon^+(ds, dy) \\ &\quad + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\bar{U}(s,y)^\gamma \bar{W}^U(ds, dy), \qquad t \geq 0, x \in \mathbb{R}. \end{aligned} \tag{A.3}$$

Let $N(t,x)$ denote the stochastic integral term in the above. The first term on the right-hand side of (A.3) can be rewritten as

$$I_1(t,x) = I_{1,\varepsilon}(t,x) = \sum_{s_i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}, s_i \leq t} \int_{\mathbb{R}} p_{t-s_i}(x-y) J_\varepsilon^{x_i}(y)\, dy \tag{A.4}$$

(the meaning of the above if $t = s_i$ some i is obvious). Recall that $x_i \in [0,1]$ and so y in the above integral may be restricted to $|y| \leq 2$. Therefore for

$s_i \le t \le T$,

$$e^{\lambda|x|} p_{t-s_i}(x-y) \le C p_{2(t-s_i)}(x-y). \tag{A.5}$$

It follows that

$$\begin{aligned}
&\sup_{t\le T, x\in\mathbb{R}} e^{\lambda|x|} I_1(t,x) \\
&\quad \le \sum_{s_i \le t-2\varepsilon} C(t-s_i)^{-1/2}\varepsilon \\
&\qquad + \sum_{t-2\varepsilon < s_i < t} \sqrt{\varepsilon} \int_{\mathbb{R}} p_{2(t-s_i)}(x-y)\,dy + \mathbf{1}(s_i = t) e^{\lambda|x|} J_\varepsilon^{x_i}(x) \qquad \text{(A.6)} \\
&\quad \le C\left[\int_0^t (t-s)^{-1/2}\,ds + \varepsilon^{1/2}\right] \\
&\quad \le C,
\end{aligned}$$

uniformly on $\varepsilon \in (0,1]$. By (A.3) and (A.6) we have for $t \le T$ and all x,

$$\begin{aligned}
E(\bar{U}(t,x)^q) &\le C[E(I_1(t,x)^q) + E(|N(t,x)|^q)] \\
&\le C[e^{-\lambda|x|} + E(|N(t,x)|^q)].
\end{aligned} \tag{A.7}$$

For $q \ge 1$ and $\lambda, t > 0$ let

$$\nu(q,\lambda,t) = \sup_{0\le s\le t} \int e^{\lambda|x|} E[\bar{U}(s,x)^q]\,dx,$$

and note that ν implicitly depends on ε. Using the Burkholder–Davis–Gundy inequality and Jensen's inequality, we get for $q \ge 2$,

$$\begin{aligned}
&E[|N(t,x)|^q] \\
&\quad \le CE\left[\left(\int_0^t \int p_{t-s}(x-y)^2 \bar{U}(s,y)^{2\gamma}\,dy\,ds\right)^{q/2}\right] \qquad \text{(A.8)} \\
&\quad \le CE\left[\int_0^t \int p_{t-s}(x-y)^2 \bar{U}(s,y)^{\gamma q}\,dy\,ds\right] \\
&\qquad \times \left(\int_0^t \int p_{t-s}(x-y)^2\,dy\,ds\right)^{(q/2)-1} \\
&\quad \le Ct^{(q-2)/4} E\left[\int_0^t \int p_{t-s}(x-y)^2 [\bar{U}(s,y)^{q/2} + \bar{U}(s,y)^q]\,dy\,ds\right].
\end{aligned}$$

The final inequality follows because $p_{t-s}(x-y)^2 \le (t-s)^{-1/2} p_{t-s}(x-y)$ and $a^{\gamma q} \le a^{q/2} + a^q$. A short calculation using the above bound, just as in the bottom display on page 349 of Mueller and Perkins (1992) shows that

$$\nu(q,\lambda,t) \le C\left[1 + \sup_{s\le t}\int e^{\lambda|x|}E(|N(t,x)|^q)\,dx\right]$$

[by (A.7) with 2λ in place of λ]

$$\le C + C\int_0^t (t-s)^{-1/2}[\nu(q/2,\lambda,s) + \nu(q,\lambda,s)]\,ds$$

$$\le C\left[1 + \nu(q/2,\lambda,t) + \int_0^t (t-s)^{-1/2}\nu(q,\lambda,s)\,ds\right].$$

A generalized Gronwall inequality [e.g., see Lemma 4.1 of Mueller and Perkins (1992)] shows that the above implies that for $q \ge 2$,

$$\nu(q,\lambda,t) \le (1+\nu(q/2,\lambda,t))\exp(4Ct^{1/2}) \qquad \text{for all } t \le T. \tag{A.9}$$

The obvious induction on $q = 2^n$ will now give (a) providing we can show

$$\nu(1,\lambda,T) \le C. \tag{A.10}$$

It follows from (A.3) and an argument using localization and Fubini's theorem that

$$\sup_{t\le T}\sup_x e^{\lambda|x|}E[\bar U(t,x)] \le \sup_{t\le T}\sup_x e^{\lambda|x|}E[I_1(t,x)] \le C,$$

the last inequality by (A.6). By optimizing over λ we get (A.10). Therefore we have proved Lemma A.1 part (a) except for one detail. To use Lemma 4.1 in Mueller and Perkins (1992) to derive (A.9) we need to know that $\nu(q,\lambda,T) < \infty$ (the bound can now depend on ε). To handle this issue one can localize just as in Mueller and Perkins (1992) using the facts that $t \to \bar U_t$ is in $D(\mathbb{R}_+, C^+_{\mathrm{rap}})$, and (from Proposition 2.1 and $\bar U = \sum_i \bar U^i$) that the jumps of $\bar U$ occur at $\{s_i\}$ with the ith jump equaling $J^{x_i} \le \sqrt{\varepsilon}$.

Turning to Lemma A.1 part (b), it suffices to consider $q > 2$. By (A.3), (A.6) and the first line of (A.8) for $t \le T$, $p = q/(q-2)$ and $p' = q/2$, we have by Hölder's inequality

$$\sup_x e^{\lambda|x|}E[\bar U(t,x)^q]$$

$$\le C\Bigg(1 + \sup_x E\Bigg[\Bigg(\int_0^t\int [p_{t-s}(x-y)^{1/p}e^{2\lambda|x|/q - 2\lambda|y|/q}]$$

$$\times [e^{2\lambda|y|/q}\bar U(s,y)^{2\gamma}]p_{t-s}(x-y)^{2-(1/p)}\,dy\,ds\Bigg)^{q/2}\Bigg]\Bigg)$$

$$\le C\left(1 + \sup_x\left(\int_0^t\int p_{t-s}(x-y)e^{2\lambda p|x|/q - 2\lambda p|y|/q}\,dy(t-s)^{-1+(1/2p)}\,ds\right)^{q/2p}\right.$$

$$
\begin{aligned}
&\quad \times E\biggl[\int_0^t \int e^{2\lambda p'|y|/q}\bar{U}(s,y)^{2\gamma p'}\,dy(t-s)^{-1+(1/2p)}\,ds\biggr]\biggr) \\
&\le C\biggl(1+\int_0^t (t-s)^{-(q+2)/(2q)}\,ds\nu(\gamma q,\lambda,t)\biggr) \\
&\le C.
\end{aligned}
$$

In the next to last line we have used Lemma 6.2 of Shiga (1994) and in the last line we have used Lemma A.1 part (a). $\square$

Proof of Lemma 6.4. It suffices to consider $\bar{U}$. Let C denote a constant depending on q and T which may change from line to line. We adapt the proof of Lemma A.3 of Mytnik, Perkins and Sturm (2006) to the white noise setting and with $\lambda = 0$.

By (A.3), (A.6) and the continuity properties of $\bar{U}$, we have

$$
\begin{aligned}
&E\Bigl[\sup_{t\le T, x\in\mathbb{R}} \bar{U}(t,x)^q\Bigr] \\
&\qquad \le C_{q,T}\biggl(1+E\biggl[\sup_{t\le T, x\in\mathbb{Q}, t\in\mathbb{Q}_+}\biggl|\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\bar{U}(s,y)^\gamma \bar{W}^U(ds,dy)\biggr|^q\biggr]\biggr).
\end{aligned}
$$

To handle the above expectation we carry out the argument in the proof of Lemma A.3 of Mytnik, Perkins and Sturm (2006) with $\lambda = 0$ and W a white noise. We take $a \in (0,1/4)$ and $q > \frac{3}{2a}$ in that work. With this choice of q, the arguments in Lemma A.3 of Mytnik, Perkins and Sturm (2006) then go through to show that the expectation in the above is at most

$$
\begin{aligned}
&C\int_0^T\int E\biggl[\biggl|\int_0^t\int (t-s)^{-a}p_{t-s}(x-y)\bar{U}(s,y)^\gamma\,d\bar{W}^U(s,y)\biggr|^q\biggr]\,dx\,dt \\
&\qquad \le C\int_0^T\int E\biggl[\biggl|\int_0^t\int (t-s)^{-2a}p_{t-s}(x-y)^2\bar{U}(s,y)^{2\gamma}\,dy\,ds\biggr|\biggr]^{q/2}\,dx\,dt \\
&\qquad \le C\int_0^T\int \biggl[\int_0^t\int (t-s)^{-2a-(1/2)}p_{t-s}(x-y)E(\bar{U}(s,y)^{q\gamma})\,dy\,ds\biggr]\,dx\,dt \\
&\qquad \le C,
\end{aligned}
$$

by Fubini, Lemma A.1 part (a) and the choice of a. This gives the result for $q > 3/2a$ and hence for all $q > 0$. $\square$

We turn next to the proof of Proposition 2.2 which is fairly standard. We follow the proof in Section 4 of Mueller and Perkins (1992), where a similar existence proof is given. The main difference is the immigration term in the present situation.

By the mild form of (2.10) we have

$$
\begin{aligned}
u_\varepsilon(t,x) &= \sum_i \int p(t-s_i, y-x) J^{x_i}(y)\mathbf{1}(t \ge s_i)\, dy \\
&\qquad - \sum_j \int p(t-t_j, y-x) J^{y_j}(y)\mathbf{1}(t \ge t_j)\, dy \\
&\qquad + \int_0^t \int p(t-s, y-x)|u_\varepsilon(s,y)|^\gamma W(ds,dy) \\
&\equiv I_{1,\varepsilon}(t,x) - I_{2,\varepsilon}(t,x) + N_\varepsilon(t,x).
\end{aligned} \tag{A.11}
$$

Now we give a modified version of Lemma 4.4 of Mueller and Perkins (1992). The only difference is that Lemma 4.4 of Mueller and Perkins (1992) deals with C^+_{rap} instead of C_{rap}, but the proof carries over with almost no change.

LEMMA A.3. *Let $\{X_n(t,\cdot) : t \ge 0, n \in \mathbb{N}\}$ be a sequence of continuous C_{rap}-valued processes. Suppose $\exists q > 0, \gamma > 2$ and $\forall T, \lambda > 0$ $\exists C = C(T,\lambda) > 0$ such that*

$$
\begin{aligned}
E[|X_n(t,x) - X_n(t',x')|^q] &\le C(|x-x'|^\gamma + |t-t'|^\gamma)e^{-\lambda|x|} \\
&\qquad \forall t,t' \in [0,T], |x-x'| \le 1, n \in \mathbb{N}.
\end{aligned} \tag{A.12}
$$

If $\{P_{X_n(0) : n \in \mathbb{N}}\}$ is tight on C_{rap}, then $\{P_{X_n} : n \in \mathbb{N}\}$ is tight on $C(\mathbb{R}_+, C_{\mathrm{rap}})$.

We also need Lemma 4.3 of Mueller and Perkins (1992):

LEMMA A.4. *If $T, \lambda > 0$ there is a constant $C(T,\lambda) < \infty$ such that*

$$
\begin{aligned}
&\int_0^t \int (p_{t-s}(y-x) - p_{t'-s}(y-x'))^2 e^{-\lambda|y|}\, dy\, ds \\
&\qquad \le C(T,\lambda)(|x-x'| + (t-t')^{1/2})e^{-\lambda|x|} \\
&\qquad\qquad \forall 0 < t' < t \le T, |x-x'| \le 1, \lambda > 0,
\end{aligned}
$$

where $p_u(z)$ is defined to be 0 if $u < 0$.

Clearly $t \to I_{\ell,\varepsilon}(t,\cdot)$ is in $D(\mathbb{R}_+, C_{\mathrm{rap}})$ with jumps only at $\{s_i\}$ for $\ell = 1$ and at $\{t_j\}$ if $\ell = 2$. It is fairly easy to see that for t, x fixed $I_{\ell,\varepsilon}(t,x)$ converges in probability to

$$
I(t,x) = \int_0^{t\wedge 1} \int_0^1 p(t-s, x-y)\, dy\, ds
$$

by the weak law of large numbers. We need convergence in path space. It is easy to check that $t \to I(t,\cdot)$ is in $C(\mathbb{R}_+, C_{\mathrm{rap}})$.

LEMMA A.5. *For $\ell = 1,2$, $I_{\ell,\varepsilon}$ converges in probability in $D(\mathbb{R}_+, C_{\mathrm{rap}})$ to I as $\varepsilon \downarrow 0$.*

PROOF. The argument is routine if a bit tedious. We sketch the proof for $\ell = 2$ where $t_j = j\varepsilon$. If $\delta = \varepsilon^{3/4}$, write

$$\begin{aligned} I_{2,\varepsilon}(t,x) &= \sum_{t_j \le t-\delta} \varepsilon \int [p_{t-t_j}(y_j - x + \sqrt{\varepsilon} w) - p_{t-t_j}(y_j - x)] J(w)\, dw \\ &\quad + \sum_{t-\delta < t_j \le t} S_{t-t_j} J_\varepsilon^{y_j}(x) + \sum_{t_j \le t-\delta} \varepsilon p_{t-t_j}(y_j - x) \\ &= T_{1,\varepsilon} + T_{2,\varepsilon} + T_{3,\varepsilon}. \end{aligned}$$

It is easy to check that for any $\lambda, T > 0$,

$$\sup_{t \le T, x \in \mathbb{R}} e^{\lambda|x|} |T_{2,\varepsilon}(t,x)| \le C_{T,\lambda} \delta / \sqrt{\varepsilon} \to 0$$

and

$$\sup_{t \le T, x \in \mathbb{R}} e^{\lambda|x|} |T_{1,\varepsilon}(t,x)| \le C_{T,\lambda} \sqrt{\varepsilon} (1 + \ln(1/\varepsilon)) \to 0.$$

So it suffices to show that $T_{3,\varepsilon}$ converges in probability in $D(\mathbb{R}_+, C_{\mathrm{rap}}^+)$ to I.

We next write

$$\begin{aligned} T_{3,\varepsilon}(t,x) &= \sum_{t_j \le t-\delta} \left(\varepsilon p_{t-t_j}(y_j - x) - \varepsilon \int_0^1 p_{t-t_j}(y - x)\, dy \right) \\ &\quad + \sum_{t_j \le t-\delta} \varepsilon \int_0^1 p_{t-t_j}(y - x)\, dy \\ &\equiv T_{4,\varepsilon} + T_{5,\varepsilon}. \end{aligned}$$

$T_{5,\varepsilon}$ is a Riemman sum for $\int_0^{t\wedge 1} \int_0^1 p_{t-s}(y-x)\, dy\, ds$ (note that $t_j \le 1$, whence the truncation by 1), and using the $t - \delta$ cut-off, the Gaussian tail and $y \in [0,1]$, it is easy to see that for any $\lambda, T > 0$,

$$\lim_{\varepsilon \to 0} \sup_{t \le T, x \in \mathbb{R}} e^{\lambda|x|} \left| T_{5,\varepsilon} - \int_0^{t\wedge 1} \int_0^1 p_{t-s}(y-x)\, dy\, ds \right| = 0.$$

Therefore it remains to show that $T_{4,\varepsilon} \to 0$ in probability in $D(\mathbb{R}_+, C_{\mathrm{rap}})$. $T_{4,\varepsilon}$ is a sum of mean 0 independent random variables, and so one easily sees that

$$E(T_{4,\varepsilon}(t,x)^2) \le \varepsilon^2 \sum_{t_j \le t-\delta} p_{2(t-t_j)}(0) \to 0 \qquad \text{as } \varepsilon \downarrow 0.$$

If we could show for any $\varepsilon_n \downarrow 0$,

$$\{T_{4,\varepsilon_n} : n\} \text{ is } C\text{-tight in } D(\mathbb{R}_+, C_{\mathrm{rap}})$$

the result would follow as the only possible weak limit point is 0 by the above.

Let $\hat{p}_{t-t_j}(y_j - x) = p_{t-t_j}(y_j - x) - \int_0^1 p_{t-t_j}(y-x)\,dy$ and

$$[t-\delta]_\varepsilon = \max\{j\varepsilon : j\varepsilon \le t-\delta, j \in \mathbb{Z}_+\}.$$

To work in the space of continuous C_{rap}-valued paths, we interpolate $T_{4,\varepsilon}$ linearly and define

$$\begin{aligned}\widetilde{T}_{4,\varepsilon_n}(t,x) = &\sum_{t_j \le [t-\delta_n]_{\varepsilon_n}} \varepsilon \hat{p}_{t-t_j}(y_j - x) \\ &+ ((t-\delta_n) - [t-\delta_n]_{\varepsilon_n})\hat{p}_{t-[t-\delta_n]_{\varepsilon_n}-\varepsilon_n}(y_{1+([t-\delta_n]_{\varepsilon_n}/\varepsilon_n)} - x),\end{aligned}$$

so that $t \to \widetilde{T}_{4,\varepsilon_n}(t,\cdot) \in C(\mathbb{R}_+, C_{\mathrm{rap}})$. If d is the metric on C_{rap}, then it is clear that

$$\lim_{n\to\infty} \sup_{t\le T} d(\widetilde{T}_{4,\varepsilon_n}(t), T_{4,\varepsilon_n}(t)) = 0 \qquad \text{for all } T > 0.$$

Therefore it remains to show that

$$\{\widetilde{T}_{4,\varepsilon_n} : n\} \text{ is tight in } C(\mathbb{R}_+, C_{\mathrm{rap}}). \tag{A.13}$$

This is proved by a straightforward application of Lemma A.3, as we illustrate below.

To illustrate the method of the aforementioned proof let us bound the spatial moments and work with $T_{4,\varepsilon}$, hence dropping the trivial continuity correction and dependence on n. Assume $0 \le t \le T$, $\lambda > 0$ and $|x - x'| \le 1$. For $q \ge 2$ we use a predictable square function inequality of Burkholder [see Theorem 21.1 of Burkholder (1973)] as follows:

$$\begin{aligned}&e^{\lambda|x|}E[|T_{4,\varepsilon}(t,x) - T_{4,\varepsilon}(t,x')|^q] \\ &\quad\le e^{\lambda|x|}c_q\Bigg[\Bigg|\sum_{t_j\le[t-\delta]_\varepsilon} \varepsilon^2 E((\hat{p}_{t-t_j}(y_j-x) - \hat{p}_{t-t_j}(y_j-x'))^2)\Bigg|^{q/2} \\ &\qquad + \sum_{t_j\le[t-\delta]_\varepsilon} \varepsilon^q E(|\hat{p}_{t-t_j}(y_j-x) - \hat{p}_{t-t_j}(y_j-x')|)^q\Bigg].\end{aligned} \tag{A.14}$$

Now for $q \ge 2$ and for, say $x > x'$,

$$\begin{aligned}&e^{\lambda|x|}E[|\hat{p}_{t-t_j}(y_j-x) - \hat{p}_{t-t_j}(y_j-x')|^q] \\ &\quad\le ce^{\lambda|x|}\int_0^1 |p_{t-t_j}(y-x) - p_{t-t_j}(y-x')|^q\,dy \\ &\quad\le C_{\lambda,T}(t-t_j)^{-1/2}\int_0^1 |p_{t-t_j}(y-x) - p_{t-t_j}(y-x')|^{q-1}\,dy.\end{aligned}$$

In the last line we used the bound on $|x-x'|$ and the fact that $y\in[0,1]$ to use the Gaussian tail of $(p_{t-t_j}(y-x)+p_{t-t_j}(y-x'))$ to absorb the $e^{\lambda|x|}$ as in (A.5). By using the spatial derivative of $p_t(z)$ and then carrying out a change of variables, we may bound the above by

$$C_{\lambda,T}(t-t_j)^{-1/2}\int_0^1 (t-t_j)^{-(q-1)/2} \times \left|\int \mathbf{1}\left(\frac{y-x}{\sqrt{t-t_j}}\le z\le \frac{y-x'}{\sqrt{t-t_j}}\right) z p_1(z)\,dz\right|^{q-1} dy$$
$$\le C_{\lambda,T}(t-t_j)^{-q+0.5}|x-x'|^{q-1}.$$

We use the above in (A.14) with $q=2$ and general q to conclude that

$$e^{|x|}E[|T_{4,\varepsilon}(t,x)-T_{4,\varepsilon}(t,x')|^q]$$
$$\le C_{\lambda,T}\left(\sum_{t_j\le[t-\delta]_\varepsilon}\varepsilon^2(t-t_j)^{-3/2}\right)^{q/2}|x-x'|^{q/2}$$
$$+C_{\lambda,T}\sum_{t_j\le[t-\delta]_\varepsilon}\varepsilon^q(t-t_j)^{-q+0.5}|x-x'|^{q-1}$$
$$\le C_{\lambda,T}|x-x'|^{q/2},$$

where we used $\delta=\varepsilon^{3/4}$, $q\ge 2$ and an elementary calculation in the last line. So taking $q>4$ gives the required spatial increment bound in Lemma A.3.

A similar, but slightly more involved, argument verifies the hypotheses of Lemma A.3 for the time increments. Here when $0\le t'-t\le\varepsilon$ the linear interpolation term must be used and the cases $[t'-\delta]_\varepsilon=[t-\delta]_\varepsilon$ and $[t'-\delta]_\varepsilon=[t-\delta]_\varepsilon+\varepsilon$ are treated separately. The details are left for the reader. This establishes (A.13) and so completes the proof. □

Next we apply Lemma A.3 to $X_n(t,x)=N_{\varepsilon_n}(t,x)$ for any $\varepsilon_n\downarrow 0$ by showing that (A.12) holds for $X_n=N_{\varepsilon_n}$.

LEMMA A.6. *$\exists q>0,\gamma>2$ and $\forall T,\lambda>0$ $\exists C=C(T,\lambda)>0$ such that*

$$E[|N_\varepsilon(t,x)-N_\varepsilon(t',x')|^q]\le C(|x-x'|^\gamma+|t-t'|^\gamma)e^{-\lambda|x|}$$
$$\forall t,t'\in[0,T],|x-x'|\le 1, 0<\varepsilon<1. \tag{A.15}$$

PROOF. Here we follow the proof of Proposition 4.5 of Mueller and Perkins (1992). Let $q\ge 1$, $\lambda>0$, $0\le t'<t\le T$ and $|x-x'|\le 1$. First, Jensen's inequality shows that for nonnegative functions f,g, we have

$$\left(\int fg\right)^q\le\left(\int f^q g\right)\left(\int g\right)^{q-1}.$$

Now using the Burkholder–Davis–Gundy inequality and Jensen's inequality and allowing c_q to vary from line to line, we find

$$
\begin{aligned}
&E[|N_\varepsilon(t,x)-N_\varepsilon(t',x')|^{2q}]\\
&\quad\le c_q E\biggl[\biggl(\int_0^t\int (p_{t-s}(y-x)-p_{t'-s}(y-x'))^2 e^{-\lambda|y|}\\
&\qquad\qquad\qquad\qquad\times e^{\lambda|y|}|u_\varepsilon(s,y)|^{2\gamma}\,dy\,ds\biggr)^q\biggr]\\
&\quad\le c_q E\biggl[\int_0^t\int |u_\varepsilon(s,y)|^{2\gamma q}e^{\lambda|y|(q-1)}(p_{t-s}(y-x)-p_{t'-s}(y-x'))^2\,dy\,ds\biggr]\\
&\qquad\times\biggl(\int_0^t\int (p_{t-s}(y-x)-p_{t'-s}(y-x'))^2 e^{-\lambda|y|}\,dy\,ds\biggr)^{q-1}\\
&\quad\le c_q E\biggl[\int_0^t\int |u_\varepsilon(s,y)|^{8\gamma q}e^{4\lambda|y|(q-1)}\,dy\,ds\biggr]^{1/4}\\
&\qquad\times\biggl(\int_0^t\int |p_{t-s}(y-x)-p_{t'-s}(y-x')|^{8/3}\,dy\,ds\biggr)^{3/4}\\
&\qquad\times C'(T,\lambda,q)(|x-x'|^{q-1}+|t-t'|^{(q-1)/2})e^{-\lambda(q-1)|x|}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(Hölder's inequality and Lemma A.4)}\\
&\quad\le C'(T,\lambda,q)(|x-x'|^{q-1}+|t-t'|^{(q-1)/2})e^{-\lambda(q-1)|x|}
\end{aligned}
$$

by Lemma A.1(a) (recall that $|u_\varepsilon|=|U_\varepsilon-V_\varepsilon|\le\bar U_\varepsilon+\bar V_\varepsilon$) and an elementary calculation. The result follows. □

Proof of Proposition 2.2. Recall that $\varepsilon_n=\frac{1}{n}$. Lemma A.6 allows us to conclude that $N_{\varepsilon_n}(t,x)$ is tight in $C(\mathbb{R}_+,C_{\mathrm{rap}})$ as $n\to\infty$. Hence by Lemma A.5 and (A.11), $\{u_{\varepsilon_n}\}$ is C-tight in $D(\mathbb{R}_+,C_{\mathrm{rap}})$.

It remains to show that any limit point satisfies equation (1.6) (it will then necessarily be a C_{rap}-valued solution). Recall from (2.10) we have

$$
\begin{aligned}
\langle u_\varepsilon(t),\phi\rangle&=\sum_i \mathbf{1}(s_i\le t)\langle J_\varepsilon^{x_i},\phi\rangle-\sum_j\mathbf{1}(t_j\le t)\langle J_\varepsilon^{y_j},\phi\rangle\\
&\quad+\int_0^t\frac12\langle u_\varepsilon(s),\Delta\phi\rangle\,ds+\int_0^t\int|u_\varepsilon(s,x)|^\gamma\phi(x)W(ds,dx)
\end{aligned}\tag{A.16}
$$

for $\phi\in C_c^\infty$.

If $\phi \in C_c(\mathbb{R})$, then a simple calculation using the strong law of large numbers shows that with probability 1,

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_i \mathbf{1}(s_i \le t)\langle J_{\varepsilon_n}^{x_i}, \phi\rangle = (t \wedge 1) \int_0^1 \phi(x)\,dx, \\
&\lim_{n\to\infty} \sum_j \mathbf{1}(t_j \le t)\langle J_{\varepsilon_n}^{y_j}, \phi\rangle = (t \wedge 1) \int_0^1 \phi(x)\,dx.
\end{aligned} \tag{A.17}
$$

It is easy to interpolate in t and conclude that the above convergence is uniform in t with probability 1. By considering a countable dense set of ϕ in $C_c(\mathbb{R})$, we may conclude that with probability 1 for all $\phi \in C_c(\mathbb{R})$ the convergence in (A.17) holds uniformly in t.

Choose a subsequence $\{n_k\}$ so that $u_{\varepsilon_{n_k}}$ converges weakly to u in $D(\mathbb{R}_+, C_{\mathrm{rap}})$ where u has continuous paths. To ease eye strain, we write u_k for $u_{\varepsilon_{n_k}}$. By Skorokhod's theorem we may change spaces so that (recall convergence in cadlag space D to a continuous path means uniform convergence on compacts)

$$
\lim_{k\to\infty} \sup_{t\le T} d(u_k(t), u(t)) = 0 \qquad \text{for all } T > 0 \text{ a.s.}
$$

This fact and the above convergence in (A.17) show that with probability 1 for all $\phi \in C_c^\infty$, the left-hand side of (A.16) and first three terms on the right-hand side of the same equation converge uniformly in t to the same terms but with u in place of u_ε, or in the case of (A.17), to the right-hand side of (A.17). Hence the last term on the right-hand side of (A.16) must also converge uniformly in t a.s. to a continuous limit $M_t(\phi)$. So for all $\phi \in C_c^\infty$ we have

$$
\langle u_t, \phi\rangle = \int_0^t \frac{1}{2}\langle u(s), \Delta\phi\rangle\,ds + M_t(\phi). \tag{A.18}
$$

We see that $M_t(\phi)$ is the a.s. limit of the stochastic integral in (A.16). Using the boundedness of the moments uniformly in ε from Lemma A.1, it is now standard to deduce that $M_t(\phi)$ is a continuous $\mathcal{F}_t$-martingale with square function $\int_0^t \int |u(s,x)|^{2\gamma}\phi(x)^2\,dx\,ds$. Here $\mathcal{F}_t$ is the right continuous filtration generated by $t \to u_t$. It is also routine to construct a white noise W, perhaps an enlarged space, so that $M_t(\phi) = \int_0^t \int u(s,x)^\gamma \phi(x)\,dW(s,x)$ for all $t \ge 0$ a.s. for all $\phi \in C_c^\infty$. Put this into (A.18) to see that u is a C_{rap}-valued solution of (1.6) and we are done. □

APPENDIX B: CONSTRUCTION OF APPROXIMATE SOLUTIONS AND PROOF OF PROPOSITION 2.1

Let us fix $\varepsilon \in (0,1]$. For this ε we construct the sequence of processes mentioned in Proposition 2.1, approximating them by a system of processes with "soft-killing." Fix $n > 0$, and define the sequence of processes $(U^{i,n}, V^{i,n}, \widetilde{U}^{i,n}, \widetilde{V}^{i,n})$ as follows. For any $\phi \in C_b^2(\mathbb{R})$, let

$$
\text{(B.1)}\quad \begin{cases}
U_t^{i,n}(\phi) = \langle J^{x_i}, \phi \rangle \mathbf{1}(t \geq s_i) \\
\qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} U^n(s,x)^{\gamma - 1/2} U^{i,n}(s,x)^{1/2} \phi(x) W^{i,n,U}(ds,dx) \\
\qquad\qquad + \displaystyle\int_0^t U_s^{i,n}\left(\frac{1}{2}\Delta\phi\right) ds - n \int_0^t \langle U_s^{i,n} V_s^n, \phi \rangle \, ds, \qquad t \geq 0, i \in \mathbb{N}_\varepsilon, \\
V_t^{j,n}(\phi) = \langle J^{y_j}, \phi \rangle \mathbf{1}(t \geq t_j) \\
\qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} V^n(s,x)^{\gamma - 1/2} V^{j,n}(s,x)^{1/2} \phi(x) W^{j,n,V}(ds,dx) \\
\qquad\qquad + \displaystyle\int_0^t V_s^{j,n}\left(\frac{1}{2}\Delta\phi\right) ds - n \int_0^t \langle V_s^{j,n} U_s^n, \phi \rangle \, ds, \qquad t \geq 0, j \in \mathbb{N}_\varepsilon, \\
\widetilde{U}_t^{i,n}(\phi) = \displaystyle\int_0^t \int_{\mathbb{R}} [(\widetilde{U}^n(s,x) + U^n(s,x))^{2\gamma} - U^n(s,x)^{2\gamma}]^{1/2} \\
\qquad\qquad\qquad \times \sqrt{\dfrac{\widetilde{U}^{i,n}(s,x)}{\widetilde{U}^n(s,x)}} \phi(x) \widetilde{W}^{i,n,U}(ds,dx) \\
\qquad\qquad + \displaystyle\int_0^t \widetilde{U}_s^{i,n}\left(\frac{1}{2}\Delta\phi\right) ds + n \int_0^t \langle U_s^{i,n} V_s^n, \phi \rangle \, ds, \qquad t \geq 0, i \in \mathbb{N}_\varepsilon, \\
\widetilde{V}_t^{j,n}(\phi) = \displaystyle\int_0^t \int_{\mathbb{R}} [(\widetilde{V}^n(s,x) + V^n(s,x))^{2\gamma} - V^n(s,x)^{2\gamma}]^{1/2} \\
\qquad\qquad\qquad \times \sqrt{\dfrac{\widetilde{V}^{j,n}(s,x)}{\widetilde{V}^n(s,x)}} \phi(x) \widetilde{W}^{j,n,V}(ds,dx) \\
\qquad\qquad + \displaystyle\int_0^t \widetilde{V}_s^{j,n}\left(\frac{1}{2}\Delta\phi\right) ds + n \int_0^t \langle V_s^{j,n} U_s^n, \phi \rangle \, ds, \qquad t \geq 0, j \in \mathbb{N}_\varepsilon,
\end{cases}
$$

where

$$
U_t^n = \sum_i U_t^{i,n}, \qquad V_t^n = \sum_j V_t^{j,n},
$$

$$
\widetilde{U}_t^n = \sum_i \widetilde{U}_t^{i,n}, \qquad \widetilde{V}_t^n = \sum_j \widetilde{V}_t^{j,n},
$$

and $\{W^{i,n,U}, W^{j,n,V}, \widetilde{W}^{k,n,U}, \widetilde{W}^{l,n,V}\}_{i,j,k,l \in \mathbb{N}_\varepsilon}$ is a collection of mutually independent white noises. For $\phi \in C_b^2(\mathbb{R})$, let $\{M_t^{i,n,U}(\phi)\}_{t \geq 0}, \{M_t^{j,n,V}(\phi)\}_{t \geq 0}$, $\{\widetilde{M}_t^{i,n,U}(\phi)\}_{t \geq 0}, \{\widetilde{M}_t^{j,n,V}(\phi)\}_{t \geq 0}$ denote the stochastic integrals on the right-hand side of the equations for $U^{i,n}, V^{j,n}, \widetilde{U}^{i,n}, \widetilde{V}^{j,n}$, respectively, in (B.1).

For each n, a solution taking values in $(C_{\text{rap}}^+)^{4N_\varepsilon}$ to the system of above equations can be constructed via standard steps by extending the procedure in Shiga (1994). We will comment further on this point below.

We also define the following nondecreasing $M_F(\mathbb{R})$-valued processes:

$$K_t^{i,n,U}(\phi) = n\int_0^t \langle U_s^{i,n} V_s^n, \phi\rangle\, ds, \qquad t \ge 0, \phi \in C_b(\mathbb{R}),$$

$$K_t^{j,n,V}(\phi) = n\int_0^t \langle V_s^{j,n} U_s^n \phi, \rangle\, ds, \qquad t \ge 0, \phi \in C_b(\mathbb{R}).$$

Clearly,

$$\sum_{i\in\mathbb{N}_\varepsilon} K_t^{i,n,U} = \sum_{j\in\mathbb{N}_\varepsilon} K_t^{j,n,V} =: K_t^n,$$

and $(U^n, V^n, \widetilde{U}^n, \widetilde{V}^n)$ satisfies the following system of equations for $\phi \in C_b^2(\mathbb{R})$:

$$\text{(B.2)}\qquad \begin{cases} U_t^n(\phi) = \displaystyle\sum_{i\in\mathbb{N}_\varepsilon} \langle J^{x_i}, \phi\rangle \mathbf{1}(t \ge s_i) \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} U^n(s,x)^\gamma \phi(x) W^{n,U}(ds,dx) \\ \qquad\qquad + \displaystyle\int_0^t U_s^n\left(\frac{1}{2}\Delta\phi\right) ds - K_t^n(\phi), \qquad t \ge 0, \\ V_t^n(\phi) = \displaystyle\sum_{j\in\mathbb{N}_\varepsilon} \langle J^{y_j}, \phi\rangle \mathbf{1}(t \ge t_j) \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} V^n(s,x)^\gamma \phi(x) W^{n,V}(ds,dx) \\ \qquad\qquad + \displaystyle\int_0^t V_s^n\left(\frac{1}{2}\Delta\phi\right) ds - K_t^n(\phi), \qquad t \ge 0, \\ \widetilde{U}_t^n(\phi) = \displaystyle\int_0^t \int_{\mathbb{R}} [(\widetilde{U}^n(s,x) + U^n(s,x))^{2\gamma} - U^n(s,x)^{2\gamma}]^{1/2} \\ \qquad\qquad\quad \times \phi(x) \widetilde{W}^{n,U}(ds,dx) \\ \qquad\qquad + \displaystyle\int_0^t \widetilde{U}_s^n\left(\frac{1}{2}\Delta\phi\right) ds + K_t^n(\phi), \qquad t \ge 0, \\ \widetilde{V}_t^n(\phi) = \displaystyle\int_0^t \int_{\mathbb{R}} [(\widetilde{V}^n(s,x) + V^n(s,x))^{2\gamma} - V^n(s,x)^{2\gamma}]^{1/2} \\ \qquad\qquad\quad \times \phi(x) \widetilde{W}^{n,V}(ds,dx) \\ \qquad\qquad + \displaystyle\int_0^t \widetilde{V}_s^n\left(\frac{1}{2}\Delta\phi\right) ds + K_t^n(\phi), \qquad t \ge 0, \end{cases}$$

with $W^{n,U}, W^{n,V}, \widetilde{W}^{n,U}, \widetilde{W}^{n,V}$ being a collection of independent space–time white noises. For $i \in \mathbb{N}_\varepsilon$, define $\bar{U}_t^{i,n} \equiv U_t^{i,n} + \widetilde{U}_t^{i,n}, \bar{V}_t^{i,n} \equiv V_t^{i,n} + \widetilde{V}_t^{i,n}, t \geq 0$ and

$$\bar{U}_t^n \equiv \sum_i \bar{U}_t^{i,n}, \qquad \bar{V}_t^n \equiv \sum_j \bar{V}_t^{j,n}, \qquad t \geq 0. \tag{B.3}$$

Since $\{W^{i,n,U}, W^{j,n,V}, \widetilde{W}^{k,n,U}, \widetilde{W}^{l,n,V}, i,j,k,l \in \mathbb{N}_\varepsilon\}$ is a collection of independent white noises, and by stochastic calculus, one can easily show that the processes $\bar{U}^n, \bar{V}^n$ satisfy equations (2.7), and so by Mytnik (1998) they have laws on $D([0,T], C_{\mathrm{rap}}^+)$ which are independent of n.

Here we comment further on the construction of $(U^{i,n}, V^{i,n}, \widetilde{U}^{i,n}, \widetilde{V}^{i,n})_{i\in\mathbb{N}_\varepsilon}$, the solution to (B.1). As we have mentioned above, one can follow the procedure indicated in the proof of Theorem 2.6 in Shiga (1994) by extending it to systems of equations. In the proof, one constructs an approximating sequence of processes $\{(U^{i,n,k}, V^{i,n,k}, \widetilde{U}^{i,n,k}, \widetilde{V}^{i,n,k})_{i\in\mathbb{N}_\varepsilon}\}_{k\geq 1}$ with globally Lipschitz coefficients, and shows that this sequence is tight in

$$\prod_{i=1}^{\mathbb{N}_\varepsilon} (C([s_i,\infty), C_{\mathrm{rap}}^+) \times C([t_i,\infty), C_{\mathrm{rap}}^+) \times C([s_i,\infty), C_{\mathrm{rap}}^+) \times C([t_i,\infty), C_{\mathrm{rap}}^+)),$$

and each limit point satisfies (B.1). The only subtle point is that the drift coefficients $U^{i,n}(\cdot)V^n(\cdot)$ and $V^{i,n}(\cdot)U^n(\cdot)$ in the system of limiting equations (B.1) do not satisfy a linear growth condition. However, note that, by (B.3), any solution to (B.1) satisfies the following bounds:

$$U^{i,n}, \widetilde{U}^{i,n}, U^n, \widetilde{U}^n \leq \bar{U}^n, \qquad V^{i,n}, \widetilde{V}^{i,n}, V^n, \widetilde{V}^n \leq \bar{V}^n, \tag{B.4}$$

where $\bar{U}^n$ and $\bar{V}^n$ have good moment bounds by Lemma 6.4. Hence, it is possible to construct $\{(U^{i,n,k}, V^{i,n,k}, \widetilde{U}^{i,n,k}, \widetilde{V}^{i,n,k})_{i\in\mathbb{N}_\varepsilon}\}_{k\geq 1}$ so that the bound in Lemma 6.4 holds uniformly in k: for any $q, T > 0$, there exists $C_{q,T}$ such that

$$\begin{aligned}\sup_{k\geq 1}\sup_{i\in\mathbb{N}_\varepsilon} E\Big[\sup_{s\leq T, x\in\mathbb{R}} (U^{i,n,k}(s,x)^q + \widetilde{U}^{i,n,k}(s,x)^q + V^{i,n,k}(s,x)^q + \widetilde{U}^{i,n,k}(s,x)^q)\Big] \\ \leq C_{q,T}.\end{aligned}$$

With this uniform bound in hand, it is not difficult to check that the moment bound (6.5) from Shiga (1994) [which is in fact (A.12) with $\lambda = 0$], holds for $\{U^{i,n,k}\}_{k\geq 1}, \{V^{i,n,k}\}_{k\geq 1}$, $\{\widetilde{U}^{i,n,k}\}_{k\geq 1}$, $\{\widetilde{V}^{i,n,k}\}_{k\geq 1}$, for all $i \in \mathbb{N}_\varepsilon$, on time intervals of the form $[\frac{(i-1)\varepsilon}{2}, \frac{i\varepsilon}{2}), i \in \mathbb{N}_\varepsilon$ and $[N_\varepsilon \varepsilon, T]$. This, in turn, by Lemma 6.3 in Shiga (1994) implies the tightness of the corresponding processes in $D^\varepsilon(\mathbb{R}_+, C_{\mathrm{tem}}^+)$. Here

$$C_{\mathrm{tem}} := \{f \in C(\mathbb{R}) : \|f\|_\lambda < \infty \text{ for any } \lambda < 0\},$$

endowed with the topology induced by the norms $\|\cdot\|_\lambda$ for $\lambda<0$, and C_{tem}^+ is the set of nonnegative functions in C_{tem}. Finally, since the limiting processes $U^{i,n}$, $\widetilde{U}^{i,n}, i\in\mathbb{N}_\varepsilon$, (resp., $V^{i,n}$, $\widetilde{V}^{i,n}, i\in\mathbb{N}_\varepsilon$) are dominated by $\bar{U}$ (resp., $\bar{V}$) in $D^\varepsilon(\mathbb{R}_+,C_{\mathrm{rap}}^+)$, it follows that $U^{i,n}$, $\widetilde{U}^{i,n}$, $V^{i,n}$, $\widetilde{V}^{i,n}, i\in\mathbb{N}_\varepsilon$, are in $D^\varepsilon(\mathbb{R}_+,C_{\mathrm{rap}}^+)$ as well. This, together with the domination (B.4) and Lemma A.1, allows us to take functions in C_{tem}^2 as test functions in (B.1); however for our purposes it will be enough to use functions from $C_b^2(\mathbb{R})$ as test functions.

Fix an arbitrary $T>1$.

REMARK B.1. In what follows we are going to show the tightness of the sequence of the processes constructed above on the time interval $[0,T]$. We will prove that limit points have the properties stated in Proposition 2.1 on $[0,T]$. Since $T>1$ is arbitrary, this argument immediately yields the claim of the theorem on the time interval $[0,\infty)$.

Define $E=[0,T]\times\mathbb{R}$. We identify a finite measure K on E with the nondecreasing path in $D([0,T],M_F(\mathbb{R}))$ given by $t\to K_t(\cdot)=K([0,t]\times\{\cdot\})$.

PROPOSITION B.2. *$\{(U^{i,n},\widetilde{U}^{i,n},V^{i,n},\widetilde{V}^{i,n},K^{i,n,U},K^{i,n,V})_{i\in\mathbb{N}_\varepsilon}\}_{n\geq1}$ is tight in $(C([0,T]\setminus\mathcal{G}_\varepsilon,M_F(\mathbb{R}))^4\times M_F(E)^2)^{N_\varepsilon}$. Moreover, any limit point $(U^i,\widetilde{U}^i,V^i,\widetilde{V}^i,K^{i,U},K^{i,V})_{i\in\mathbb{N}_\varepsilon}$ has the following properties:*

(1) *$U^i,\widetilde{U}^i,V^i,\widetilde{V}^i\in C([0,T]\setminus\mathcal{G}_\varepsilon,C_{\mathrm{rap}}^+)\cap D^\varepsilon([0,T],L^1(\mathbb{R})),\forall i\in\mathbb{N}_\varepsilon$;*
(2) *$K^{i,U},K^{i,V}\in D^\varepsilon([0,T],M_F(\mathbb{R})),\forall i\in\mathbb{N}_\varepsilon$;*
(3) *$(U^i,\widetilde{U}^i,V^i,\widetilde{V}^i,K^{i,U},K^{i,V})_{i\in\mathbb{N}_\varepsilon}$ satisfy (2.1)–(2.4).*

The above proposition is the key for proving Proposition 2.1. The proposition will be proved via a series of lemmas.

LEMMA B.3. *$\{K^n\}_{n\geq1}$ is tight in $M_F(E)$, and $\{K_T^n(1)\}_{n\geq1}$ is $L^1(dP)$-bounded.*

PROOF. First note that by rewriting equation (2.7) for $\bar{U}^n$ in the mild form [see (A.3)] one can easily get that for any $\phi\in C_b^+(\mathbb{R})$,

$$E[\bar{U}_t^n(\phi)]\leq E\biggl[\sum_{s_i\in\mathcal{G}_\varepsilon^{\mathrm{odd}},s_i\leq t}\int_{\mathbb{R}}\int_{\mathbb{R}}p_{t-s_i}(z-y)J_\varepsilon^{x_i}(y)\phi(z)\,dy\,dz\biggr]$$

$$=\sum_{s_i\in\mathcal{G}_\varepsilon^{\mathrm{odd}},s_i\leq t}\int_0^1\int_{\mathbb{R}}\int_{\mathbb{R}}p_{t-s_i}(z-y)J_\varepsilon^{x}(y)\phi(z)\,dy\,dz\,dx \tag{B.5}$$

$$= \sum_{s_i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}, s_i \le t} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^1 p_{t-s_i}(z-y) J_\varepsilon^x(y) \phi(z)\, dx\, dz\, dy.$$

Estimating the above integrals, we have

$$\begin{aligned} E[\bar{U}_t^n(\phi)] &\le \varepsilon \sum_{s_i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}, s_i \le t} \int_{\mathbb{R}} S_{t-s_i}\phi(y) \mathbf{1}(|y| \le 2)\, dy \\ &\le \sup_{s \le t} \int_{\mathbb{R}} S_s \phi(y) \mathbf{1}(|y| \le 2)\, dy, \end{aligned}$$

where $\{S_t\}_{t \ge 0}$ is the Brownian semigroup corresponding to the transition density function $\{p_t(x), t \ge 0, x \in \mathbb{R}\}$.

For any nonnegative $\phi \in C_b^2(\mathbb{R})$ we have from (B.2),

$$\begin{aligned} E[K_t^n(\phi)] &\le E\left[\sum_{i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}} \int_{\mathbb{R}} J_\varepsilon^{x_i}(y)\phi(y)\, dy\right] + E\left[\int_0^t U_s^n\left(\left|\frac{\Delta\phi}{2}\right|\right) ds\right] \\ &\le \sum_{i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}} \int_0^1 \int_{\mathbb{R}} J_\varepsilon^x(y)\phi(y)\, dy\, dx + E\left[\int_0^t \bar{U}_s^n\left(\left|\frac{\Delta\phi}{2}\right|\right) ds\right] \qquad \text{(B.6)} \\ &\le \int_{\mathbb{R}} \mathbf{1}(|y| \le 2)\phi(y)\, dy + \int_0^t \sup_{r \le s} \int_{\mathbb{R}} S_r\left(\left|\frac{\Delta\phi}{2}\right|\right)(y) \mathbf{1}(|y| \le 2)\, dy\, ds. \end{aligned}$$

Now by taking $\phi = 1$ we get that the sequence of the total masses $\{K_T^n(1)\}_{n \ge 1}$ is bounded in $L^1(dP)$. Moreover for any $\delta > 0$ we can choose $R > 3$ sufficiently large and ϕ such that $\phi(z) = 0$ for $|z| \le R - 1, \phi(z) = 1$ for $|z| \ge R$ with the property that

$$S_t\left(\left|\frac{\Delta\phi}{2}\right|\right)(y) \le \delta \qquad \forall t \in [0, T], y \in [-2, 2].$$

This shows that

$$E\left[\int_{|z| \ge R} K_T^n(dz)\right] \le E[K_T^n(\phi)] \le 4T\delta \qquad \forall n \ge 1,$$

by (B.6), and our choice of ϕ and R. This, in turn, together with the $L^1(dP)$-boundedness of total masses $\{K_T^n(1)\}_{n \ge 1}$, implies tightness of $\{K^n\}_{n \ge 1}$ in $M_F(E)$. $\square$

COROLLARY B.4. *$\{K^{i,n,U}\}_{n \ge 1}$ and $\{K^{i,n,V}\}_{n \ge 1}$ are tight in $M_F(E)$ for any $i \in \mathbb{N}_\varepsilon$.*

PROOF. The assertion follows immediately from the bound

$$K^{n,i,U}, K^{n,i,V} \le K^n \qquad \forall n \ge 1, i \in \mathbb{N}_\varepsilon. \qquad \square$$

Before we start dealing with tightness of $\{(U^n, V^n, \widetilde{U}^n, \widetilde{V}^n, K^n)\}_{n\geq 1}$ we need to introduce a lemma that will be frequently used.

LEMMA B.5. *We have:*

(a) *Let $\{W^n\}_{n\geq 1}$ be a sequence of $\{\mathcal{F}_t^n\}_{t\geq 0}$-adapted space–time white noises, and $\{b^n(t,x,\omega)\}_{n\geq 1}$ be a sequence of $\{\mathcal{F}_t^n\}_{t\geq 0}$-predictable $\times$ Borel measurable processes such that*

$$\sup_{n\geq 1}\sup_{x\in\mathbb{R}}\sup_{t\in[0,T]} E[|b^n(t,x,\cdot)|^p] < \infty \qquad \text{for some } p > 4. \tag{B.7}$$

Then the sequence of processes $\{X^n(t,x), t\in[0,T], x\in\mathbb{R}\}_{n\geq 1}$ defined by

$$X^n(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) b^n(s,y,\cdot) W^n(ds,dy), \qquad t\in[0,T], x\in\mathbb{R},$$

have versions which are tight in $C([0,T], C_{\mathrm{tem}})$.

(b) *Let W be an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted space–time white noise, and $b(t,x,\omega)$ be an $\{\mathcal{F}_t\}_{t\geq 0}$-predictable $\times$ Borel measurable process such that*

$$\sup_{x\in\mathbb{R}}\sup_{t\in[0,T]} E[|b(t,x,\cdot)|^p] < \infty \qquad \text{for some } p > 4. \tag{B.8}$$

Then the process X defined by

$$X(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) b(s,y,\cdot) W^n(ds,dy), \qquad t\in[0,T], x\in\mathbb{R},$$

has a version in $C([0,T], C_{\mathrm{tem}})$. If moreover, $|X(t,x)| \leq |\widetilde{X}(t,x)|$ for some $\widetilde{X}\in D([0,T], C_{\mathrm{rap}})$, then $X\in C([0,T], C_{\mathrm{rap}})$.

PROOF. (a) This assertion follows immediately from the estimates on increments of a stochastic integral [see, e.g., step 2 in the proof of Theorem 2.2 of Shiga (1994), page 432] and then an application of Lemmas 6.2 and 6.3(ii) from Shiga (1994).

(b) This again follows by using the estimates on increments of a stochastic integral [see again step 2 in the proof of Theorem 2.2 of Shiga (1994), page 432] and then applying Lemmas 6.2 and 6.3(i) in Shiga (1994), to get that the process is in $C([0,T], C_{\mathrm{tem}})$. The last assertion is obvious. □

LEMMA B.6. *Let*

$$w^n = U^n - V^n, \qquad n\geq 1.$$

Then $\{w^n\}_{n\geq 1}$ is tight in $D([0,T], C_{\mathrm{rap}})$, and every limit point is in $D^{\varepsilon}([0,T], C_{\mathrm{rap}})$.

PROOF. By writing the equation for w^n in mild form we get

$$\begin{aligned} w^n(t,x) = &\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)(\eta_\varepsilon^+(ds,dy) - \eta_\varepsilon^-(ds,dy)) \\ &+ \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) U^n(s,y)^\gamma W^{n,U}(ds,dy) \\ &- \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) V^n(s,y)^\gamma W^{n,V}(ds,dy), \qquad t \geq 0, x \in \mathbb{R}. \end{aligned}$$

Clearly, by the definition of $\eta_\varepsilon^+, \eta_\varepsilon^-$, the first term, $I(t,x)$ (being independent of n) is tight in $D([0,T], C_{\text{rap}})$, and is in $D^\varepsilon([0,T], C_{\text{rap}})$. Using the domination

$$U^n \leq \bar{U}^n \in D([0,T], C_{\text{rap}}^+), \qquad V^n \leq \bar{V}^n \in D([0,T], C_{\text{rap}}^+), \tag{B.9}$$

and Lemmas 6.4 and B.5(a), the stochastic integral terms are tight in $C([0,T], C_{\text{tem}})$. If $S^n(t,x)$ is the difference of the above stochastic integral terms, then the domination

$$|S^n(t,x)| \leq \bar{U}^n(t,x) + \bar{V}^n(t,x) + |I(t,x)| \in D^\varepsilon([0,T], C_{\text{rap}}^+),$$

and the definition of the norms on C_{tem} and C_{rap} shows that $\{S^n\}$ is tight in $C([0,T], C_{\text{rap}})$. $\square$

Now we are ready to deal with the tightness of $\{(U^n, V^n, \widetilde{U}^n, \widetilde{V}^n, K^n)\}_{n \geq 1}$. Let $L^p(E)$ denote the usual L^p space with respect to Lebesgue measure on E.

LEMMA B.7. *The following assertions hold:*

(a) $\{(U^n, V^n, \widetilde{U}^n, \widetilde{V}^n, K^n)\}_{n \geq 1}$ *is tight in* $L^p(E)^4 \times M_F(E)$ *for any* $p \geq 1$. *Moreover any limit point has a version*

$$(U, V, \widetilde{U}, \widetilde{V}, K) \in D^\varepsilon([0,T], C_{\text{rap}}^+)^4 \times D^\varepsilon([0,T], M_F(\mathbb{R})).$$

(b)

$$t \mapsto \int_0^t \int_{\mathbb{R}} p_{t-s}(\cdot - y) K(ds,dy) \in D^\varepsilon([0,T], C_{\text{rap}}).$$

(c) $\{K^n\}_{n \geq 1}$ *is also tight in* $C([0,T] \setminus \mathcal{G}_\varepsilon, M_F(\mathbb{R}))$, *and any of its limit points satisfies*

$$\mathbf{\Delta} K_t(1) \leq \varepsilon \qquad \forall t \in [0,T].$$

PROOF. (a) We will give the proof just for the tightness of $\{(U^n, V^n, K^n)\}_{n \geq 1}$ and the properties of its limit points, since the corresponding results for $\{(\widetilde{U}^n, \widetilde{V}^n)\}_{n \geq 1}$ and its limit points will follow along the same lines.

Recall the domination (B.9), where the laws of the upper bounds are independent of n. By this domination we immediately get that

$$\{(U^n(s,x)\,dx\,ds, V^n(s,x)\,dx\,ds)\}_{n\geq 1}$$

is tight in $(M_F(E)\times M_F(E))$. Recall also that by Lemma B.3, $\{K^n\}_{n\geq 1}$ is tight in $M_F(E)$. This, the fact that the laws of $\bar{U}_n$, $\bar{V}_n$ are independent of n, and Lemma B.6 allows us to choose a convergent subsequence of $(U^n, V^n, K^n, w^n, \bar{U}^n, \bar{V}^n)$ in $M_F(E)^3 \times D([0,T], C_{\mathrm{rap}})^3$. For simplicity of notation, we will again index this subsequence by n. Denote the corresponding limit point by $(U, V, K, w, \bar{U}, \bar{V})$.

Now, for any $\phi \in C_b(\mathbb{R})$, let

$$M_t^{n,U}(\phi) \equiv \int_0^t \int_{\mathbb{R}} U^n(s,x)^\gamma \phi(x) W^{n,U}(ds,dx), \qquad t\in[0,T],$$

$$M_t^{n,V}(\phi) \equiv \int_0^t \int_{\mathbb{R}} V^n(s,x)^\gamma \phi(x) W^{n,V}(ds,dx), \qquad t\in[0,T],$$

denote the martingales given by the stochastic integrals in the semimartingale decomposition (B.2) for $U_t^n(\phi)$ and $V_t^n(\phi)$. For any $\phi\in C_b(\mathbb{R})$, use the Burkholder–Davis–Gundy inequality, and again the domination (B.9), to get, that for any $p\geq 2, \lambda > 0$,

$$\begin{aligned} &E[|M_t^{n,U}(\phi) - M_u^{n,U}(\phi)|^p] \\ &\quad \leq C_p \sup_{s\leq T, x\in\mathbb{R}} e^{(\lambda p/2)|x|} E[\bar{U}(s,x)^{p\gamma}] \\ &\qquad \times \left[\int_{\mathbb{R}} e^{-\lambda|x|}|\phi(x)|^2\,dx\right]^{p/2} (t-u)^{p/2}, \\ &\qquad\qquad \forall 0\leq u\leq t\leq T. \end{aligned} \tag{B.10}$$

This, together with Lemma A.1(b) and Kolmogorov's tightness criterion, implies that

$$\{M_\cdot^{n,U}(\phi)\}_{n\geq 1} \text{is tight in } C([0,T],\mathbb{R}) \tag{B.11}$$

for any $\phi\in C_b(\mathbb{R})$. Similarly,

$$\{M_\cdot^{n,V}(\phi)\}_{n\geq 1} \text{is tight in } C([0,T],\mathbb{R}) \tag{B.12}$$

for any $\phi\in C_b(\mathbb{R})$. Let $\mathcal{D}$ be a countable subset of $C_b^2(\mathbb{R})$ which is bounded-pointwise dense in $C_b(\mathbb{R})$. That is, the smallest class containing $\mathcal{D}$ and closed under bounded pointwise limits contains $C_b(\mathbb{R})$. By the above, we can take a further subsequence, which for simplicity we will index again by n, so that all the sequences of martingales $\{M_\cdot^{n,U}(\phi)\}_{n\geq 1}$, $\{M_\cdot^{n,V}(\phi)\}_{n\geq 1}$ indexed by functions ϕ from $\mathcal{D}$, converge in $C([0,T],\mathbb{R})$. For $\phi\in\mathcal{D}$, we will denote the limiting processes by $M^U_\cdot(\phi), M^V_\cdot(\phi)$, respectively. Now let us switch to a probability space where

$$(U^n, V^n, K^n, w^n, \bar{U}^n, \bar{V}^n) \to (U, V, K, w, \bar{U}, \bar{V}) \quad \text{in } M_F(E)^3 \times D([0,T], C_{\mathrm{rap}})^3,$$
$$(M^{n,U}(\phi_1), M^{n,V}(\phi_2)) \to (M^U(\phi_1), M^V(\phi_2)) \quad \text{in } C([0,T], \mathbb{R})^2 \ \forall \phi_1, \phi_2 \in \mathcal{D}, \tag{B.13}$$

as $n \to \infty$, a.s.

In our next step, we will verify convergence of $\{(U^n, V^n)\}_{n\geq 1}$ in $L^p(E)^2$, for any $p \geq 1$. First, by $L^1(dP)$-boundedness of the total mass of K^n (Lemma B.3), we have

$$nE\left[\int_0^T \int_{\mathbb{R}} U^n_s(x) V^n_s(x)\, dx\, ds\right] = E[K^n_T(1)] \leq C, \tag{B.14}$$

uniformly in n for some constant C. Therefore we get

$$E\left[\int_0^T \int_{\mathbb{R}} U^n_s(x) V^n_s(x)\, dx\, ds\right] \to 0 \qquad \text{as } n \to \infty, \tag{B.15}$$

and hence

$$\int_0^T \int_{\mathbb{R}} (U^n_s(x) \wedge V^n_s(x))^2\, dx\, ds \to 0, \tag{B.16}$$

in $L^1(dP)$. By taking another subsequence if necessary, we may assume

$$(U^n_s(x) \wedge V^n_s(x)) \to 0 \qquad \text{in } L^2(E), P\text{-a.s.}$$

Now recall again the domination

$$U^n \leq \bar{U}^n \to \bar{U} \qquad \text{in } D([0,T], C^+_{\mathrm{rap}}), P\text{-a.s.},$$

which implies that for any $p \geq 1$,

$$(U^n_s(x) \wedge V^n_s(x)) \to 0 \qquad \text{in } L^p(E), P\text{-a.s.}$$

Also by

$$U^n_t(x) = (U^n_s(x) \wedge V^n_s(x)) + (w^n_t(x))^+,$$

we get that in fact

$$U^n \to (w)^+ \qquad \text{in } L^p(E), \text{ for any } p \geq 1, P\text{-a.s.}, \tag{B.17}$$

and hence $U(dt, dx) = w_t(x)^+\, dt\, dx$. With some abuse of notation we denote the density of $U(dt, dx)$ by $U_t(x)$. Similarly we get

$$V(dt, dx) = w_t(x)^-\, dt\, dx,$$

and we denote its density by $V_t(x)$. In what follows we will use the continuous in space versions of the densities of $U(dt,dx), V(dt,dx)$, that is, $U_t(x) = w_t(x)^+, V_t(x) = w_t(x)^-$, and hence, by Lemma B.6, we get that $(U,V) \in D^\varepsilon([0,T], C_{\mathrm{rap}})^2$. We delay the proof of the assertion that $K \in D^\varepsilon([0,T], M_F(\mathbb{R}))$ until the proof of part (b).

(b) Fix an arbitrary $\phi \in \mathcal{D}$. We will go to the limit in (B.2) for $\{U^n_\cdot(\phi)\}_{n\geq 1}$. As $\{U^n\}_{n\geq 1}$ converges a.s. to w^+ in $L^2(ds,dx)$, and

$$U^n \leq \bar{U}^n \to \bar{U} \qquad \text{in } D([0,T], C_{\mathrm{rap}}),$$

it is easy to see that $\{U^n_\cdot(\phi)\}_{n\geq 1}$ converges to $w^+_\cdot(\phi) \equiv \int w^+_\cdot(x)\phi(x)\,dx$ in $L^2[0,T]$ a.s. As for the right-hand side, use (B.17) with $p=1$ to get

$$\sup_{t\leq T}\left|\int_0^t U^n_s\left(\frac{1}{2}\Delta\phi\right)ds - \int_0^t U_s\left(\frac{1}{2}\Delta\phi\right)ds\right| \leq \|U^n - U\|_{L^1(E)}\|\Delta\phi/2\|_\infty \to 0.$$

In particular this implies that $\{\int_0^\cdot U^n_s(\frac{1}{2}\Delta\phi)\,ds\}_{n\geq 1}$ converges to $\int_0^\cdot U_s(\frac{1}{2}\Delta\phi)\,ds$ in $C([0,T],\mathbb{R})$ (and hence in $L^2[0,T]$). By (a) $\{K^n(\phi)(ds)\}_{n\geq 1}$ converges to $K(\phi)(ds)$ as finite signed measures on $[0,T]$ a.s., and therefore $\{K^n_\cdot(\phi)\}_{n\geq 1}$ converges in $L^2[0,T]$ to $K_\cdot(\phi)$ a.s. Since the immigration term does not change with n, it also converges in $L^2[0,T]$.

Now we have to deal with convergence of the stochastic integral term, that we denoted by $M^{n,U}(\phi)$. We proved in (a) that $\{M^{n,U}(\phi)\}_{n\geq 1}$ converges a.s. in $C([0,T],\mathbb{R})$. Moreover, by (B.10), the martingales $M^{n,U}_t(\phi)$ are bounded in $L^p(dP)$ uniformly in n and $t\in[0,T]$, for all $p\geq 2$, and hence the limiting process is a continuous martingale that we will call $M^U(\phi)$. Turning to its quadratic variation, it follows from (B.17) that the sequence $\{(U^n)^{2\gamma}\}_{n\geq 1}$ converges to $U^{2\gamma}$ in $L^2(E)$ a.s. and this implies that

$$\begin{aligned}\langle M^{n,U}_\cdot(\phi)\rangle_t &= \int_0^t \int_{\mathbb{R}} U^n(s,x)^{2\gamma}\phi(x)^2\,dx\,ds \\ &\to \int_0^t\int_{\mathbb{R}} U(s,x)^{2\gamma}\phi(x)^2\,dx\,ds \qquad \text{as } n\to\infty, P\text{-a.s.}\end{aligned} \tag{B.18}$$

Hence, again by boundedness of $M^{n,U}_t(\phi)$ in $L^p(dP), p\geq 2$, uniformly in $t\in[0,T], n\geq 1$, we get that the limiting continuous martingale M^U has quadratic variation

$$\langle M^U_\cdot(\phi)\rangle_t = \int_0^t\int_{\mathbb{R}} U(s,x)^{2\gamma}\phi(x)^2\,dx\,ds$$

for any $\phi\in\mathcal{D}$. Since $\mathcal{D}$ is bounded-pointwise dense in $C_b(\mathbb{R})$, M^U can be extended to a martingale measure on E, and one can show by standard procedure that there is a space–time white noise W^U such that

$$M_t^U(\phi) = \int_0^t \int_{\mathbb{R}} U(s,x)^\gamma \phi(x) W^U(ds,dx), \qquad t \in [0,T], \forall \phi \in C_b(\mathbb{R}).$$

Now we are ready to take limits in (B.2) in $L^2([0,T])$. We get

$$\begin{aligned} U_t(\phi) = {} & \sum_i \langle J^{x_i}, \phi \rangle \mathbf{1}(t \geq s_i) \\ & + \int_0^t \int_{\mathbb{R}} U(s,x)^\gamma \phi(x) W^U(ds,dx) \\ & + \int_0^t U_s\left(\frac{1}{2}\Delta\phi\right) ds - K_t(\phi), \qquad t \in [0,T]. \end{aligned} \tag{B.19}$$

Note that although some of the convergences leading to the above equation hold in $L^2[0,T]$, all terms are right continuous in t and so the equality holds for all t, and not just for a.e. t. By equation (B.19) and the fact that $U \in D^\varepsilon([0,T], C_{\mathrm{rap}})$ [from (a)] we see that $K_\cdot(\phi) \in D^\varepsilon([0,T],\mathbb{R})$. It then follows from $K \in M_F(E)$ that $K_\cdot \in D^\varepsilon([0,T], M_F(\mathbb{R}))$, and this proves the last part of (a).

Now we will rewrite the above equation in the mild form. The derivation is a bit more complicated than, for example, (A.3) for $\bar{U}$, due to the presence of the measure-valued term K. For any $\phi \in C_b^+(\mathbb{R})$, $t \in [0,T] \setminus \mathcal{G}_\varepsilon$,

$$\begin{aligned} U_t(\phi) = {} & \sum_{s_i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}, s_i \leq t} \int_{\mathbb{R}} S_{t-s_i}\phi(y) J_\varepsilon^{x_i}(y)\, dy \\ & + \int_0^t \int_{\mathbb{R}} S_{t-s}\phi(y) U(s,y)^\gamma W^U(ds,dy) \\ & - \int_0^t \int_{\mathbb{R}} S_{t-s}\phi(y) K(ds,dy). \end{aligned}$$

Writing S_t in terms of p_t, we have

$$\begin{aligned} U_t(\phi) = {} & \int_{\mathbb{R}} \phi(x) \sum_{s_i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}, s_i \leq t} \int_{\mathbb{R}} p_{t-s_i}(y-x) J_\varepsilon^{x_i}(y)\, dy\, dx \\ & + \int_{\mathbb{R}} \phi(x) \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) U(s,y)^\gamma W^U(ds,dy)\, dx \\ & - \int_{\mathbb{R}} \phi(x) \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) K(ds,dy)\, dx, \qquad P\text{-a.s.}, \end{aligned} \tag{B.20}$$

where the last equality follows by the Fubini and the stochastic Fubini theorems. Note that we take the time t outside the set $\mathcal{G}_\varepsilon$ since, for $t \in \mathcal{G}_\varepsilon$,

$K(\{t\},dx)$ could be strictly positive, and with p_0 being a delta measure, this creates difficulties with applying the Fubini theorem. Therefore the case of $t\in\mathcal{G}_\varepsilon$ will be treated separately.

By (a), we know that

$$U\in D^\varepsilon([0,T],C^+_{\mathrm{rap}}),\qquad P\text{-a.s.} \tag{B.21}$$

By the domination

$$U^\gamma\le \bar{U}^\gamma\in D^\varepsilon([0,T],C^+_{\mathrm{rap}}),$$

Lemma 6.4, and Lemma B.5(b) we may choose a version of the stochastic integral so that

$$t\mapsto\int_0^t\int_{\mathbb{R}}p_{t-s}(\cdot-y)U(s,y)^\gamma W^U(ds,dy)\in C([0,T],C_{\mathrm{rap}}),\qquad P\text{-a.s.,} \tag{B.22}$$

and in what follows we will always consider such a version. This, and the fact that $K.\in D^\varepsilon([0,T],M_F(\mathbb{R}))$, implies that the equality in (B.20) holds P-a.s. *for all* $t\in[0,T]\setminus\mathcal{G}_\varepsilon$, and, hence, we get

$$\begin{aligned}U_t(x)={}&\sum_{s_i\in\mathcal{G}^{\mathrm{odd}}_\varepsilon,s_i\le t}\int_{\mathbb{R}}p_{t-s_i}(x-y)J^{x_i}_\varepsilon(y)\,dy\\&+\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)U(s,y)^\gamma W^U(ds,dy)\\&-\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)K(ds,dy),\\&\text{Leb-a.e. } x\in\mathbb{R},\text{for each } t\in([0,T]\setminus\mathcal{G}_\varepsilon)\ ,P\text{-a.s.}\end{aligned} \tag{B.23}$$

Now let us check that the above equation holds for all $(t,x)\in([0,T]\setminus\mathcal{G}_\varepsilon)\times\mathbb{R}$, P-a.s. [recall again that Lemma B.5(b) is used to select an appropriate jointly continuous version of the stochastic integral]. First, note that the steps similar to those leading to (B.23) easily imply

$$\begin{aligned}U_t(x)={}&S_{t-r}U_r(x)+\sum_{s_i\in\mathcal{G}^{\mathrm{odd}}_\varepsilon,r<s_i\le t}\int_{\mathbb{R}}p_{t-s_i}(x-y)J^{x_i}_\varepsilon(y)\,dy\\&+\int_r^t\int_{\mathbb{R}}p_{t-s}(x-y)U(s,y)^\gamma W^U(ds,dy)\\&-\int_r^t\int_{\mathbb{R}}p_{t-s}(x-y)K(ds,dy),\\&\text{Leb-a.e. } x\in\mathbb{R},\text{for all } r,t\in[0,T]\setminus\mathcal{G}_\varepsilon,r\le t,\ P\text{-a.s.}\end{aligned} \tag{B.24}$$

Lemma B.5(b) could be easily strengthened to assure, that, in fact, the process

$$X(r,t,x) \equiv \int_r^t \int_{\mathbb{R}} p_{t-s}(x-y)U(s,y)^\gamma W^U(ds,dy), \qquad 0 \le r \le t \le T, x \in \mathbb{R}, \text{is } P\text{-a.s. continuous in } (r,t,x) \tag{B.25}$$

and

$$X(t,t,\cdot) = 0 \qquad \forall t \in [0,T]. \tag{B.26}$$

Again, to be more precise, there exists just a version of the process X such that (B.25) holds, and, in what follows, we will always consider such a version.

As was already noted following Lemma A.4,

$$t \mapsto \sum_{s_i \in \mathcal{G}_\varepsilon^{\mathrm{odd}}, s_i \le t} \int_{\mathbb{R}} p_{t-s_i}(\cdot - y) J_\varepsilon^{x_i}(y)\,dy \in D^\varepsilon([0,T], C^+_{\mathrm{rap}}), \qquad P\text{-a.s.} \tag{B.27}$$

Let us take $A \subset \Omega$ such that $P(A) = 1$ and for each $\omega \in A$, (B.21) and (B.23)–(B.27) hold. Fix an arbitrary $\omega \in A$ and $(t,x) \in ((0,T] \setminus \mathcal{G}_\varepsilon) \times \mathbb{R}$. Then choose $\{(r_l, z_k)\}_{l,k \ge 1}$ such that the equality in (B.24) holds with (r_l, t, z_k) in place of (r,t,x), and $(r_l, z_k) \to (t,x) \in ([0,T] \setminus \mathcal{G}_\varepsilon) \times \mathbb{R}$, as $l, k \to \infty$. Also assume that $r_l < t$, for all $l \ge 1$. Note that both $\{(r_l, z_k)\}_{l,k \ge 1}, (t,x)$ may depend on ω. We would like to show

$$\lim_{k \to \infty} \int_0^t \int_{\mathbb{R}} p_{t-s}(z_k - y) K(ds,dy) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) K(ds,dy). \tag{B.28}$$

Fix $\delta > 0$. By (B.21), (B.25) and (B.26) we can choose l^* sufficiently large so that, with $r^* \equiv r_{l^*}$, we have

$$|U_t(z_k) - S_{t-r^*} U_{r^*}(z_k)| + \left| \int_{r^*}^t \int_{\mathbb{R}} p_{t-s}(z_k - y) U(s,y)^\gamma W^U(ds,dy) \right| \le \delta \tag{B.29}$$

for all $k \ge 1$. Note that we assume without loss of generality that

$$[r^*, t] \subset [0,T] \setminus \mathcal{G}_\varepsilon.$$

Now we are ready to show (B.28). First, by the bounded convergence theorem and $K_\cdot \in D^\varepsilon([0,T], M_F(\mathbb{R}))$, we get

$$\int_0^{r^*} \int_{\mathbb{R}} p_{t-s}(z_k - y) K(ds,dy) \to \int_0^{r^*} \int_{\mathbb{R}} p_{t-s}(x-y) K(ds,dy) \tag{B.30}$$

as $k \to \infty$. Next consider (B.24) with $r = r^*$, $x = z_k$, to conclude that

$$\begin{aligned} U_t(z_k) &= S_{t-r^*}U_{r^*}(z_k) \\ &\quad + \int_{r^*}^t \int_{\mathbb{R}} p_{t-s}(z_k - y) U(s,y)^\gamma W^U(ds, dy) \\ &\quad - \int_{r^*}^t \int_{\mathbb{R}} p_{t-s}(z_k - y) K(ds, dy) \qquad \forall k \ge 1. \end{aligned} \tag{B.31}$$

Therefore,

$$\begin{aligned} &\int_{r^*}^t \int_{\mathbb{R}} p_{t-s}(z_k - y) K(ds, dy) \\ &\quad \le |U_t(z_k) - S_{t-r^*}U_{r^*}(z_k)| + \left| \int_{r^*}^t \int_{\mathbb{R}} p_{t-s}(z_k - y) U(s,y)^\gamma W^U(ds, dy) \right| \\ &\quad \le \delta \qquad \forall k \ge 1, \end{aligned} \tag{B.32}$$

where the last bound follows from (B.29). This together with Fatou's lemma and $K \in D^\varepsilon([0,T], M_F(\mathbb{R}))$ implies

$$\begin{aligned} &\int_{r^*}^t \int_{\mathbb{R}} p_{t-s}(x - y) K(ds, dy) \\ &\quad \le \liminf_{k\to\infty} \int_{r^*}^t \int_{\mathbb{R}} p_{t-s}(z_k - y) K(ds, dy) \le \delta. \end{aligned} \tag{B.33}$$

Equations (B.32), (B.33) and (B.30) imply

$$\limsup_{k\to\infty} \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y) K(ds, dy) - \int_0^t \int_{\mathbb{R}} p_{t-s}(z_k - y) K(ds, dy) \right| \le 3\delta,$$

and since δ was arbitrary, (B.28) follows.

Equation (B.28) together with (B.21), (B.22), (B.27) implies that the equality in (B.23) holds for *all* $(t,x) \in ([0,T] \setminus \mathcal{G}_\varepsilon) \times \mathbb{R}$ on a set of full probability measure. Moreover, since all the other terms in (B.23) except $\int_0^t \int_{\mathbb{R}} p_{t-s}(\cdot - y) K(ds, dy)$ are in $D^\varepsilon([0,T], C^+_{\mathrm{rap}})$, we get that, in fact,

$$t \mapsto \int_0^t \int_{\mathbb{R}} p_{t-s}(\cdot - y) K(ds, dy) \in C([0,T] \setminus \mathcal{G}_\varepsilon, C^+_{\mathrm{rap}}), \qquad P\text{-a.s.}$$

Now let $t \in \mathcal{G}_\varepsilon$, and let us show that, at t, the C^+_{rap}-valued mapping $r \mapsto \int_0^r \int_{\mathbb{R}} p_{r-s}(\cdot - y) K(ds, dy)$ is right continuous and with a left limit. We will prove it for $t = s_j \in \mathcal{G}^{\mathrm{odd}}_\varepsilon$ for some j (for $t \in \mathcal{G}^{\mathrm{even}}_\varepsilon$ the argument is the same, even simpler). Note that the measure $K(\{s_j\}, dx)$ is absolutely continuous with respect to Lebesgue measure. This follows from (B.19) and the fact that U is in $D^\varepsilon([0,T], C^+_{\mathrm{rap}})$. We will denote the density of $K(\{s_j\}, dx)$ by

$K(\{s_j\},x), x\in\mathbb{R}$. Take $\eta>0$ sufficiently small such that $(s_j,s_j+\eta]\subset[0,T]\setminus\mathcal{G}_\varepsilon$. Then, since (B.23) holds for all $(t,x)\in([0,T]\setminus\mathcal{G}_\varepsilon)\times\mathbb{R}$, we get

$$\begin{aligned}U_{s_j+\eta}(x) &= \sum_{s_i\in\mathcal{G}_\varepsilon^{\mathrm{odd}},s_i<s_j}\int_{\mathbb{R}} p_{s_j+\eta-s_i}(x-y)J_\varepsilon^{x_i}(y)\,dy \\ &\quad+\int_{\mathbb{R}} p_\eta(x-y)J_\varepsilon^{x_j}(y)\,dy \\ &\quad+\int_0^{s_j+\eta}\int_{\mathbb{R}} p_{s_j+\eta-s}(x-y)U(s,y)^\gamma W^U(ds,dy) \\ &\quad-\int_0^{s_j+\eta}\int_{\mathbb{R}} p_{s_j+\eta-s}(x-y)(K(ds,dy)-\delta_{s_j}(ds)K(\{s_j\},dy)) \\ &\quad-\int_{\mathbb{R}} p_\eta(x-y)K(\{s_j\},y)\,dy \qquad \forall x\in\mathbb{R}.\end{aligned}\tag{B.34}$$

Take $\eta\downarrow 0$. Since the measure $(K(ds,dy)-\delta_{s_j}(ds)K(\{s_j\},dy))$ gives zero mass to the set $\{s_j\}\times\mathbb{R}$, by the argument similar to the one used in the case of $t\in[0,T]\setminus\mathcal{G}_\varepsilon$, we can easily derive that

$$\begin{aligned}&\int_0^{s_j+\eta}\int_{\mathbb{R}} p_{s_j+\eta-s}(\cdot-y)(K(ds,dy)-\delta_{s_j}(ds)K(\{s_j\},dy)) \\ &\qquad\to\int_0^{s_j}\int_{\mathbb{R}} p_{s_j-s}(\cdot-y)(K(ds,dy)-\delta_{s_j}(ds)K(\{s_j\},dy)),\end{aligned}$$

in C_{rap}, as $\eta\downarrow 0$. Moreover, $U_{s_j+\eta}(\cdot)$ and the first three terms on the right-hand side of (B.34) converge in C_{rap}. This immediately implies that the last term $\int_{\mathbb{R}} p_\eta(\cdot-y)K(\{s_j\},y)\,dy$ also converges in C_{rap}, and clearly the limit is

$$K(\{s_j\},\cdot)\in C_{\mathrm{rap}},\tag{B.35}$$

or more precisely a C_{rap}-valued version of this density. All together we get that (B.23) holds also for $t\in\mathcal{G}_\varepsilon^{\mathrm{odd}}$ with p_0 being the Dirac measure; moreover the C_{rap}-valued mapping $r\mapsto\int_0^r\int_{\mathbb{R}} p_{r-s}(\cdot-y)K(ds,dy)$ is right continuous at $t\in\mathcal{G}_\varepsilon^{\mathrm{odd}}$. The existence of left-hand limits for $r\mapsto\int_0^r\int_{\mathbb{R}} p_{r-s}(\cdot-y)K(ds,dy)$ at $t\in\mathcal{G}_\varepsilon^{\mathrm{odd}}$ follows by a similar argument. As we noted above, the same proof works for $t\in\mathcal{G}_\varepsilon^{\mathrm{even}}$, and this finishes the proof of (b).

(c) By the above $t\mapsto K_t$ is continuous on $[0,T]\setminus\mathcal{G}_\varepsilon$. Since $\{K^n\}$ is a sequence of continuous, nondecreasing measure-valued processes, its tightness in $M_F(E)$ immediately implies tightness on all the open intervals between the jumps of the limiting process, in the space of continuous measure-valued paths, that is, in $C([0,T]\setminus\mathcal{G}_\varepsilon,M_F(\mathbb{R}))$.

So, the only jumps K may possibly have are at the points $s_i,t_i\in\mathcal{G}_\varepsilon$. We recall that a jump of measure-valued process K at any $t\in[0,T]$ equals

$K(\{t\},dx)=K(\{t\},x)\,dx$, where by (B.35) $K(\{t\},\cdot)\in C_{\mathrm{rap}}$ for all $t\in\mathcal{G}_\varepsilon$. We now calculate the sizes of those jumps. Consider the possible jump at s_i. Assume ϕ is a nonnegative function in $C_c^2(\mathbb{R})$. By (B.19) (and it's analogue for V), $U=w^+$ and $V=w^-$, we have the following conditions on $w^{\pm}_{s_i}$:

$$\boldsymbol{\Delta}\langle w^+,\phi\rangle(s_i)=\langle J^{x_i},\phi\rangle-\langle K(\{s_i\},\cdot),\phi\rangle, \tag{B.36}$$

$$\boldsymbol{\Delta}\langle w^-,\phi\rangle(s_i)=-\langle K(\{s_i\},\cdot),\phi\rangle\le 0. \tag{B.37}$$

The above are preserved under bounded pointwise limits in ϕ and so continue to hold for any bounded Borel $\phi\ge 0$.

We consider two cases. First assume ϕ is such that

$$\mathrm{supp}(\phi)\subset\{x: w^-_{s_i-}(x)=0\}.$$

Then $\boldsymbol{\Delta}\langle w^-,\phi\rangle(s_i)=\langle w^-_{s_i},\phi\rangle\ge 0$ and so (B.37) immediately implies that $\langle K(\{s_i\},\cdot),\phi\rangle=0$.

Now let ϕ be such that

$$\mathrm{supp}(\phi)\subset\{x: w^+_{s_i-}(x)=0\}.$$

Then $\boldsymbol{\Delta}\langle w^+_{s_i},\phi\rangle=\langle w^+_{s_i},\phi\rangle\ge 0$ and so (B.36) immediately implies that $\langle K(\{s_i\},\cdot),\phi\rangle\le\langle J^{x_i},\phi\rangle$.

We may write $1=\phi_1+\phi_2$, where ϕ_i is as in case i ($i=1,2$) [because $w^+_{s_i-}(x)w^-_{s_i-}(x)\equiv 0$]. It therefore follows that

$$\boldsymbol{\Delta}\langle K_{s_i},1\rangle=\langle K(\{s_i\},\cdot),1\rangle\le\langle J^{x_i},1\rangle=\varepsilon,$$

and we are done. $\square$

LEMMA B.8. *The following assertions hold.*

(a) *For any $i\in\mathbb{N}_\varepsilon$, $\{U^{i,n}\}_{n\ge1}$, $\{\widetilde{U}^{i,n}\}_{n\ge1}$, $\{V^{i,n}\}_{n\ge1}$, $\{\widetilde{V}^{i,n}\}_{n\ge1}$ are tight in $C([0,T]\setminus\mathcal{G}_\varepsilon,M_F(\mathbb{R}))$.*

(b) *For any $i,j\in\mathbb{N}_\varepsilon$, and $\phi_l\in C_b(\mathbb{R}), l=1,\dots,4$,*

$$\{(M^{i,n,U}(\phi_1),M_t^{j,n,V}(\phi_2),\widetilde{M}_t^{i,n,U}(\phi_3),\widetilde{M}_t^{j,n,V}(\phi_4))\}_{n\ge1}$$

is tight in $C([0,T],\mathbb{R})^4$.

PROOF. Fix an arbitrary $i\in\mathbb{N}_\varepsilon$. Let us first prove the tightness for $\{U^{i,n}\}_{n\ge1}$. By the nonnegativity of $U^{i,n}$'s and the domination $U^{i,n}\le\bar{U}^n\to\bar{U}\in D([0,T],C^+_{\mathrm{rap}})$ a.s. [recall (B.13)], by Jakubowski's theorem [see, e.g., Theorem II.4.1 in Perkins (2002)], it is enough to prove tightness of $\{U^{i,n}(\phi)\}_{n\ge1}$ in $C([0,T]\setminus\mathcal{G}_\varepsilon,\mathbb{R})$, for any $\phi\in C_b^2(\mathbb{R})$. From (B.1) we get

$$\begin{aligned}U_t^{i,n}(\phi)&=\langle J^{x_i},\phi\rangle\mathbf{1}(t\ge s_i)+M_t^{i,n,U}(\phi)\\&\quad+\int_0^t U_s^{i,n}(\Delta\phi/2)\,ds-K_t^{i,n,U}(\phi),\qquad t\in[0,T].\end{aligned}\tag{B.38}$$

For any $p>2$, we use Hölder's inequality to bound the pth moment of the increment of the third term on the right-hand side of (B.38),

$$
\begin{aligned}
&E\left[\left|\int_u^t U_s^{i,n}\left(\frac{1}{2}\Delta\phi\right)ds\right|^p\right]\\
&\quad\le \sup_{s\le T,x\in\mathbb{R}} e^{\lambda p|x|}E[\bar{U}^n(s,x)^p]\left[\int_{\mathbb{R}} e^{-\lambda|x|}\left|\frac{1}{2}\Delta\phi(x)\right|dx\right]^p(t-u)^p, \qquad \text{(B.39)}\\
&\qquad\qquad \forall 0\le u\le t.
\end{aligned}
$$

Now use Lemma A.1(b) and the Kolmogorov tightness criterion to see that

$$
\left\{\int_0^\cdot U_s^{i,n}\left(\frac{1}{2}\Delta\phi\right)ds\right\}_{n\ge1} \qquad \text{is tight in } C([0,T],\mathbb{R}), \forall\phi\in C_b^2(\mathbb{R}). \tag{B.40}
$$

As for the martingale $M_\cdot^{i,n,U}(\phi)$, we can argue exactly as in the proof of tightness for $\{M^{n,U}(\phi)\}_{n\ge1}$ in Lemma B.7(a), by using again the domination, $U^{i,n}(s,\cdot)\le U^n(s,\cdot)\le\bar{U}^n(s,\cdot), s\in[0,T]$, to show that

$$
\{M_\cdot^{i,n,U}(\phi)\}_{n\ge1} \qquad \text{is tight in } C([0,T],\mathbb{R}) \tag{B.41}
$$

for any $\phi\in C_b(\mathbb{R})$. As for $K^{i,n,U}$, it is dominated from the above by K^n and by Lemma B.7(c), $\{K^n\}_{n\ge1}$ is tight in $C([0,T]\setminus\mathcal{G}_\varepsilon, M_F(\mathbb{R}))$. Therefore $\{K^{i,n,U}\}_{n\ge1}$ is also tight in the same space.

We combine this with (B.40), (B.41) and (B.38) to finish the proof of tightness of $\{U^{i,n}\}_{n\ge1}$ in $C([0,T]\setminus\mathcal{G}_\varepsilon, M_F(\mathbb{R}))$.

As for $\{\widetilde{U}^{i,n}\}_{n\ge1}$, we get by the same argument as above that

$$
\left\{\int_0^\cdot \widetilde{U}_s^{i,n}(\Delta\phi/2)\,ds\right\}_{n\ge1} \qquad \text{is tight in } C([0,T],\mathbb{R}), \forall\phi\in C_b^2(\mathbb{R}). \tag{B.42}
$$

For the martingale term, fix an arbitrary $\phi\in C_b$. We have again tightness of $\{\widetilde{M}_\cdot^{i,n,U}(\phi)\}_{n\ge1}$ in $C([0,T],\mathbb{R})$ by the same method as for $\{M_\cdot^{i,n,U}(\phi)\}_{n\ge1}$, by using the domination,

$$
[(\widetilde{U}^n(s,\cdot)+U^n(s,\cdot))^{2\gamma}-U^n(s,\cdot)^{2\gamma}]^{1/2}\sqrt{\frac{\widetilde{U}^{i,n}(s,\cdot)}{\widetilde{U}^n(s,\cdot)}}\le\bar{U}^n(s,\cdot)^\gamma, \qquad s\in[0,T].
$$

The tightness of $\{V^{j,n}(\phi)\}_{n\ge1}$ and $\{\widetilde{V}^{j,n}(\phi)\}_{n\ge1}$ follows in exactly the same way. $\square$

In what follows we take any converging subsequence of the processes from Lemmas B.8(a), B.7(a) and Corollary B.4. Recall that $\mathcal{D}$ is the countable subset of $C_b^2(\mathbb{R})$ which is bounded-pointwise dense in $C_b(\mathbb{R})$. By Lemma B.8(b) we can take a further subsequence, if needed, so that all the martingales from Lemma B.8(b) indexed by functions from $\mathcal{D}$ converge in $C([0,T],\mathbb{R})$.

To simplify notation we will still index this subsequence by n. Let us also switch to the Skorohod space where all the processes mentioned in the previous paragraph converge a.s. Since $(\bar U^n, \bar V^n)$ has the same law as the weakly unique in $D^\varepsilon([0,T], C^+_{\mathrm{rap}})^2$ solution to (2.7) [by Theorem 1.1 of Mytnik (1998)], we may, and shall, assume that on our probability space $(\bar U^n, \bar V^n) \to (\bar U, \bar V)$ in $D([0,T], C^+_{\mathrm{rap}})^2$, a.s., and, of course,

$$\begin{aligned} &U^{i,n}, \widetilde U^{i,n}, U^n, \widetilde U^n \le \bar U^n \qquad \forall n \ge 1, i \in \mathbb{N}_\varepsilon,\\ &V^{i,n}, \widetilde V^{i,n}, V^n, \widetilde V^n \le \bar V^n \qquad \forall n \ge 1, i \in \mathbb{N}_\varepsilon. \end{aligned} \tag{B.43}$$

For $i \in \mathbb{N}_\varepsilon$, let

$$U, V, \widetilde U, \widetilde V, \bar U, \bar V, K, U^i, V^i, \widetilde U^i, \widetilde V^i, K^{i,U}, K^{i,V}$$

be the limiting points of $\{U^n\}_{n\ge1}, \{V^n\}_{n\ge1}$, $\{\widetilde U^n\}_{n\ge1}$, $\{\widetilde V^n\}_{n\ge1}$, $\{\bar U^n\}_{n\ge1}$, $\{\bar V^n\}_{n\ge1}$, $\{K^n\}_{n\ge1}$, $\{U^{i,n}\}_{n\ge1}$, $\{V^{i,n}\}_{n\ge1}$, $\{\widetilde U^{i,n}\}_{n\ge1}$, $\{\widetilde V^{i,n}\}_{n\ge1}$, $\{K^{i,n,U}\}_{n\ge1}$, $\{K^{i,n,V}\}_{n\ge1}$, respectively. Clearly w.p.1 for all $t \in [0,T] \setminus \mathcal{G}_\varepsilon$,

$$U_t = \sum_{i\in\mathbb{N}_\varepsilon} U^i_t, \qquad \widetilde U_t = \sum_{i\in\mathbb{N}_\varepsilon} \widetilde U^i_t, \tag{B.44}$$

$$V_t = \sum_{i\in\mathbb{N}_\varepsilon} V^i_t, \qquad \widetilde V_t = \sum_{i\in\mathbb{N}_\varepsilon} \widetilde V^i_t, \tag{B.45}$$

by the corresponding equations for the approximating processes,

$$\bar U_t = U_t + \widetilde U_t, \qquad \bar V_t = V_t + \widetilde V_t \qquad \text{for all } t \in [0,T]$$

by the same reasoning and Lemma B.7(a), and

$$K = \sum_{i\in\mathbb{N}_\varepsilon} K^{i,U} = \sum_{j\in\mathbb{N}_\varepsilon} K^{j,V}.$$

By Lemma B.7(a) we may take versions of $U, \widetilde U, V, \widetilde V, \bar U, \bar V$ in $D^\varepsilon([0,T], C^+_{\mathrm{rap}})$. We next refine the state space of the subprocesses corresponding to the individual clusters.

LEMMA B.9. *For any* $i \in \mathbb{N}_\varepsilon$,

$$\begin{aligned} &(U^i, \widetilde U^i, V^i, \widetilde V^i, K^{i,U}, K^{i,V})\\ &\qquad \in (D^\varepsilon([0,T], M_F(\mathbb{R})) \cap L^2(E))^4 \times D([0,T], M_F(\mathbb{R}))^2 \end{aligned}$$

and $(U^i, \widetilde U^i, V^i, \widetilde V^i, K^{i,U}, K^{i,V})_{i\in\mathbb{N}_\varepsilon}$ *satisfy (2.1), (2.2) and (2.4).*

PROOF. Although U^i (and similarly $V^i, \widetilde U^i, \widetilde V^i$) is defined as a limit point of $\{U^{i,n}\}_{n\ge1}$ in $C([0,T] \setminus \mathcal{G}_\varepsilon, M_F(\mathbb{R}))$, it can be also considered as a limit of $\{U^{i,n}\}_{n\ge1}$ in the weak $L^2(E)$ topology [in the sequel we denote the space $L^2(E)$ equipped with the weak topology, by $L^{2,w}(E)$]. Indeed, since by (B.43), all $U^{i,n},\widetilde{U}^{i,n}$ (resp., $V^{i,n},\widetilde{V}^{i,n}$) are bounded from above by $\bar{U}^n \to \bar{U}$ in $D([0,T],C^+_{\mathrm{rap}})$ [resp., $\bar{V}^n \to \bar{V}$ in $D([0,T],C^+_{\mathrm{rap}})$], we get that, in fact,

$$\{U^{i,n}\}_{n\geq 1},\{\widetilde{U}^{i,n}\}_{n\geq 1},\{V^{i,n}\}_{n\geq 1},\{\widetilde{V}^{i,n}\}_{n\geq 1}$$

are all relatively compact in $L^{2,w}(E)$. This and the convergence of $\{U^{i,n}\}_{n\geq 1}$, $\{V^{i,n}\}_{n\geq 1}$, $\{\widetilde{U}^{i,n}\}_{n\geq 1}$, $\{\widetilde{V}^{i,n}\}_{n\geq 1}$, in $C([0,T]\setminus\mathcal{G}_\varepsilon, M_F(\mathbb{R}))$ as $n\to\infty$, imply that

$$(U^{i,n},\widetilde{U}^{i,n},V^{i,n},\widetilde{V}^{i,n}) \to (U^i,\widetilde{U}^i,V^i,\widetilde{V}^i) \qquad \text{in } L^{2,w}(E)^4, P\text{-a.s., as } n\to\infty.$$

Therefore we have

$$U^i,\widetilde{U}^i,V^i,\widetilde{V}^i \in C([0,T]\setminus\mathcal{G}_\varepsilon, M_F(\mathbb{R}))\cap L^2(E).$$

From our earlier remark prior to Proposition B.2 and $K^{i,U},K^{u,V}\in M_F(E)$, we have

$$(K^{i,U},K^{i,V})\in D([0,T],M_F(\mathbb{R}))^2.$$

Now let us derive the semimartingale decomposition for U^i. Consider the convergence of the right-hand side of the equation for $U^{i,n}(\phi)$ in (B.1). By convergence of $\{U^{i,n}\}_{n\geq 1}$ in $L^{2,w}(E)$ and in $C([0,T]\setminus\mathcal{G}_\varepsilon, M_F(\mathbb{R}))$ we get that, for any $\phi\in C_b^2(\mathbb{R})$ and any $t\leq T$,

$$\int_0^t\int_{\mathbb{R}} U_s^{i,n}(x)\frac{\Delta}{2}\phi(x)\,dx\,ds \to \int_0^t\int_{\mathbb{R}} U_s^{i}(x)\frac{\Delta}{2}\phi(x)\,dx\,ds \qquad \text{as } n\to\infty. \tag{B.46}$$

Now fix an arbitrary $\phi\in\mathcal{D}$. By Lemma B.8(b) we may assume that $M^{i,n,U}(\phi)$ converges a.s. in $C([0,T],\mathbb{R})$. Moreover, using a bound analogous to (B.10), one can immediately get that, for any $p\geq 2$, the martingale $M_t^{i,n,U}(\phi)$ is bounded in $L^p(dP)$ uniformly in n and $t\in[0,T]$. Hence, the limiting process is a continuous L^2-martingale that we will call $M^{i,U}(\phi)$. For its quadratic variation, recall that the sequence $\{(U^n)^{2\gamma-1}\}_{n\geq 1}$ converges to $U^{2\gamma-1}$ *strongly* in $L^2(E)$ [by (B.17)] and this together with convergence of $\{U^{i,n}\}_{n\geq 1}$ in $L^{2,w}(E)$ implies that, for any $\phi\in C_b(\mathbb{R})$ and $t\leq T$, w.p.1

$$\begin{aligned}\langle M^{i,n,U}(\phi)\rangle_t &= \int_0^t\int_{\mathbb{R}} U^n(s,x)^{2\gamma-1}U^{i,n}(s,x)\phi(x)^2\,dx\,ds\\ &\to \int_0^t\int_{\mathbb{R}} U(s,x)^{2\gamma-1}U^{i}(s,x)\phi(x)^2\,dx\,ds \qquad \text{as } n\to\infty.\end{aligned} \tag{B.47}$$

Hence, again by boundedness of $M_t^{i,n,U}(\phi)$, in $L^p(dP), p\geq 2$, uniformly in $t\in[0,T], n\geq 1$, we get that the limiting continuous martingale $M^{i,U}$ has quadratic variation

$$\langle M^{i,U}(\phi)\rangle_t = \int_0^t \int_{\mathbb{R}} U(s,x)^{2\gamma-1} U^i(s,x)\phi(x)^2\,dx\,ds$$

for all $\phi \in \mathcal{D} \subset C_b(\mathbb{R})$. Moreover, by repeating the above argument for $V^{i,n}$ we get that $(U^i, V^i)_{i\in\mathbb{N}_\varepsilon}$, solves the following martingale problem:

$$\text{(B.48)}\quad \begin{cases} \text{For all } \phi_i, \psi_j \in \mathcal{D} \subset C_b^2(\mathbb{R}), \\ U_t^i(\phi_i) = \langle J^{x_i}, \phi_i\rangle \mathbf{1}(t \ge s_i) + M_t^{i,U}(\phi_i) \\ \qquad\qquad + \displaystyle\int_0^t U_s^i\left(\frac{1}{2}\Delta\phi_i\right) ds - K_t^{i,U}(\phi_i) \qquad \forall t \in [0,T], i \in \mathbb{N}_\varepsilon, \\ V_t^j(\psi_j) = \langle J^{y_j}, \psi_j\rangle \mathbf{1}(t \ge t_j) + M_t^{j,V}(\psi_j) \\ \qquad\qquad + \displaystyle\int_0^t V_s^j\left(\frac{1}{2}\Delta\psi_j\right) ds - K_t^{j,V}(\psi_j) \qquad \forall t \in [0,T], j \in \mathbb{N}_\varepsilon, \end{cases}$$

where $M^{i,U}(\phi_i), M^{j,V}(\psi_j)$ are martingales such that for all $i,j \in \mathbb{N}$,

$$\text{(B.49)}\begin{cases} \langle M_\cdot^{i,U}(\phi_i), M_\cdot^{j,U}(\phi_j)\rangle_t = \delta_{i,j}\displaystyle\int_0^t \int_{\mathbb{R}} U(s,x)^{2\gamma-1} U^i(s,x)\phi_i(x)^2\,dx\,ds, \\ \langle M_\cdot^{i,V}(\psi_i), M_\cdot^{j,V}(\psi_j)\rangle_t = \delta_{i,j}\displaystyle\int_0^t \int_{\mathbb{R}} V(s,x)^{2\gamma-1} V^i(s,x)\psi_i(x)^2\,dx\,ds, \\ \langle M_\cdot^{i,U}(\phi_i), M_\cdot^{j,V}(\psi_j)\rangle_t = 0 \qquad \forall i,j \in \mathbb{N}_\varepsilon. \end{cases}$$

Note that the equality in (B.48) holds for any t in $[0,T] \setminus \mathcal{G}_\varepsilon$ since both left- and right-hand sides are continuous processes on $[0,T] \setminus \mathcal{G}_\varepsilon$; moreover the right-hand side is cadlag on $[0,T]$. Using this and the domination $U_t^i \le \bar{U}_t$ and $V_t^i \le \bar{V}_t$ for $t \notin \mathcal{G}_\varepsilon$, we may construct versions of U^i and V^i in $D^\varepsilon([0,T], M_F(\mathbb{R})) \cap L^2(E)$ so that equality in (B.48) holds for all t in $[0,T]$. Clearly the martingale problem (B.48) can be also extended to all $\phi_i, \psi_j \in C_b^2(\mathbb{R})$ by a limiting procedure, again using the $L^p(dP)$ boundedness of the martingales for any $p \ge 2$.

Now let us handle the processes $(\widetilde{U}^i, \widetilde{V}^i), i \in \mathbb{N}_\varepsilon$. By the same steps that were used to treat $(U^i, V^i)_{i\in\mathbb{N}_\varepsilon}$ we get that $(\widetilde{U}^i, \widetilde{V}^i)_{i\in\mathbb{N}_\varepsilon}$ satisfies the following martingale problem:

$$\text{(B.50)}\quad \begin{cases} \text{For all } \phi_i, \psi_j \in \mathcal{D} \subset C_b^2(\mathbb{R}), \\ \widetilde{U}_t^i(\phi_i) = \langle J^{x_i}, \phi_i\rangle \mathbf{1}(t \ge s_i) + \widetilde{M}_t^{i,U}(\phi_i) \\ \qquad\qquad + \displaystyle\int_0^t \widetilde{U}_s^i\left(\frac{1}{2}\Delta\phi_i\right) ds + K_t^{i,U}(\phi_i) \qquad \forall t \in [0,T], i \in \mathbb{N}_\varepsilon, \\ \widetilde{V}_t^j(\psi_j) = \langle J^{y_j}, \psi_j\rangle \mathbf{1}(t \ge t_j) + \widetilde{M}_t^{j,V}(\psi_j) \\ \qquad\qquad + \displaystyle\int_0^t \widetilde{V}_s^j\left(\frac{1}{2}\Delta\psi_j\right) ds + K_t^{j,V}(\psi_j) \qquad \forall t \in [0,T], j \in \mathbb{N}_\varepsilon, \end{cases}$$

where by Lemma B.8 $\widetilde{M}^{i,U}(\phi_i), \widetilde{M}^{j,V}(\psi_j)$ are continuous processes. By the same argument as before [the uniform in n and t, boundedness $L^p(dP), p \geq 2$, of the approximating martingales] they are martingales and we would like to show that, for any $i, j \in \mathbb{N}_\varepsilon$,

$$\text{(B.51)}\qquad \begin{cases} \langle \widetilde{M}^{i,U}_\cdot(\phi_i), \widetilde{M}^{j,U}_\cdot(\phi_j)\rangle_t \\ \qquad = \delta_{i,j} \displaystyle\int_0^t \int_{\mathbb{R}} \frac{(\widetilde{U}(s,x) + U(s,x))^{2\gamma} - U(s,x)^{2\gamma}}{\widetilde{U}(s,x)} \\ \qquad\qquad \times \widetilde{U}^i(s,x)\phi_i(x)^2\,dx\,ds, \\ \langle \widetilde{M}^{i,V}_\cdot(\psi_i), \widetilde{M}^{j,V}_\cdot(\psi_j)\rangle_t \\ \qquad = \delta_{i,j} \displaystyle\int_0^t \int_{\mathbb{R}} \frac{(\widetilde{V}(s,x) + V(s,x))^{2\gamma} - V(s,x)^{2\gamma}}{\widetilde{V}(s,x)} \\ \qquad\qquad \times \widetilde{V}^i(s,x)\psi_i(x)^2\,dx\,ds, \\ \langle \widetilde{M}^{i,U}_\cdot(\phi_i), \widetilde{M}^{j,V}_\cdot(\psi_j)\rangle_t = 0. \end{cases}$$

As before, the orthogonality of the limiting martingales follows easily by the uniform in n and t, $L^p(dP), p \geq 2$, boundedness of the approximating martingales and their orthogonality. Next we calculate the quadratic variations. We will do it just for $\widetilde{M}^{i,U}_\cdot(\phi)$, for some $i \in \mathbb{N}_\varepsilon$. It is enough to show that for any $\phi \in C_b(\mathbb{R})$ and $t \in [0,T]$,

$$\text{(B.52)}\qquad \begin{aligned} &\int_0^t \int_{\mathbb{R}} \frac{(\widetilde{U}^n(s,x) + U^n(s,x))^{2\gamma} - U^n(s,x)^{2\gamma}}{\widetilde{U}^n(s,x)} \widetilde{U}^{i,n}(s,x)\phi(x)\,dx\,ds \\ &\quad \to \int_0^t \int_{\mathbb{R}} \frac{(\widetilde{U}(s,x) + U(s,x))^{2\gamma} - U(s,x)^{2\gamma}}{\widetilde{U}(s,x)} \widetilde{U}^i(s,x)\phi(x)\,dx\,ds, \end{aligned}$$

in $L^1(dP)$, as $n \to \infty$. Denote

$$F(\tilde{u}, u) \equiv (\tilde{u} + u)^{2\gamma} - u^{2\gamma}.$$

Then, for any $\phi \in C_b(\mathbb{R})$ and $t \in [0,T]$, we get

$$\text{(B.53)}\qquad \begin{aligned} &\biggl| \int_0^t \int_{\mathbb{R}} \frac{F(\widetilde{U}^n(s,x), U^n(s,x))}{\widetilde{U}^n(s,x)} \widetilde{U}^{i,n}(s,x)\phi(x)\,dx\,ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \frac{F(\widetilde{U}(s,x), U(s,x))}{\widetilde{U}(s,x)} \widetilde{U}^i(s,x)\phi(x)\,dx\,ds \biggr| \\ &\qquad \leq \biggl| \int_0^t \int_{\mathbb{R}} \biggl(\frac{F(\widetilde{U}^n(s,x), U^n(s,x))}{\widetilde{U}^n(s,x)} - \frac{F(\widetilde{U}(s,x), U(s,x))}{\widetilde{U}(s,x)} \biggr) \\ &\qquad\qquad \times \widetilde{U}^{i,n}(s,x)\phi(x)\,dx\,ds \biggr| \end{aligned}$$

$$+\left|\int_0^t\int_{\mathbb{R}}\frac{F(\widetilde{U}(s,x),U(s,x))}{\widetilde{U}(s,x)}(\widetilde{U}^i(s,x)-\widetilde{U}^{i,n}(s,x))\phi(x)\,dx\,ds\right|$$
$$\equiv I^{1,n}+I^{2,n}.$$

Clearly

$$\frac{F(\widetilde{U}(s,x),U(s,x))}{\widetilde{U}(s,x)}\leq 2\gamma\bar{U}^{2\gamma-1}\in L^2(E), \tag{B.54}$$

and hence by convergence of $\widetilde{U}^{i,n}$ to $\widetilde{U}^i$ in $L^{2,w}(E)$, a.s., we get that

$$I^{2,n}\to 0 \qquad \text{as } n\to\infty \text{ a.s.}$$

and by dominated convergence it is easy to get that, in fact, the convergence is in $L^1(dP)$. As for $I^{1,n}$, by using $|\frac{\widetilde{U}^{i,n}(s,x)}{\widetilde{U}^n(s,x)}|\leq 1$ we immediately get that

$$I^{1,n}\leq\int_0^t\int_{\mathbb{R}}|F(\widetilde{U}^n(s,x),U^n(s,x))-F(\widetilde{U}(s,x),U(s,x))|\phi(x)\,dx\,ds$$
$$+\int_0^t\int_{\mathbb{R}}\frac{F(\widetilde{U}(s,x),U(s,x))}{\widetilde{U}(s,x)}|\widetilde{U}(s,x)-\widetilde{U}^n(s,x)|\phi(x)\,dx\,ds.$$

We again use (B.54) and convergence of $\widetilde{U}^n$ and U^n to $\widetilde{U}$ and U, respectively, in $L^p(E)$ for any $p\geq 1$, we immediately get that, $I^{1,n}\to 0$, a.s., as $n\to\infty$. Use again the dominated convergence theorem to get that, in fact, the convergence holds in $L^1(dP)$, and (B.52) follows. As a result we get that $(U^i,V^i,\widetilde{U}^i,\widetilde{V}^i), i\in\mathbb{N}_\varepsilon$ solves the martingale problem (B.48), (B.49), (B.50), (B.51), with all martingales corresponding to different processes being orthogonal.

Now, as before, [see the proof of Lemma B.7(b)], the martingales in the martingale problem can be represented as stochastic integrals with respect to independent white noises, and hence one immediately gets that $(U^i,V^i,\widetilde{U}^i,\widetilde{V}^i)_{i\in\mathbb{N}_\varepsilon}$ solves (2.1), (2.2) and (2.4) but with $(U^i,V^i,\widetilde{U}^i,\widetilde{V}^i)\in(D^\varepsilon([0,T],M_F(\mathbb{R}))\cap L^2(E))^4, i\in\mathbb{N}_\varepsilon$. Here we note that equality in (B.44) as $M_F(\mathbb{R})$-valued processes extends to all $t\in[0,T]$ by right-continuity. □

To finish the proof of Proposition B.2 we next verify the following lemma.

LEMMA B.10. $U^i,\widetilde{U}^i,V^i,\widetilde{V}^i\in C([0,T]\setminus\mathcal{G}_\varepsilon,C^+_{\mathrm{rap}})\cap D^\varepsilon([0,T],L^1(\mathbb{R}))$, $\forall i\in\mathbb{N}_\varepsilon$.

PROOF. We will prove it just for U^i, as the proof for the other terms goes along exactly along the same lines. Similarly to the steps in the proof of Lemma B.7(b), we first write the equation for U^i in the mild form to get

$$
\begin{aligned}
U^i(t,x) &= \int_{\mathbb{R}} p_{t-s_i}(x-y)J_\varepsilon^{x_i}(y)\,dy \\
&\quad + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)U(s,y)^{\gamma-1/2}U^i(s,y)^{1/2}W^{i,U}(ds,dy) \\
&\quad - \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)K^{i,U}(ds,dy)
\end{aligned} \tag{B.55}
$$

$$\text{Leb-a.e. } (t,x) \in ([0,T]\setminus \mathcal{G}_\varepsilon)\times\mathbb{R}.$$

We now argue as in the proof of part (b) of Lemma B.7. The first term on the right-hand side of (B.55) clearly belongs to $D^\varepsilon([0,T],C_{\mathrm{rap}})$. Similarly by the bound

$$U^{\gamma-1/2}(U^i)^{1/2} \le \bar{U}^\gamma \in D([0,T],C_{\mathrm{rap}}^+),$$

Lemma 6.4, and Lemma B.5(b), we see that the second term on the right-hand side is in $C([0,T],C_{\mathrm{rap}})$. As for the third term on the right-hand side, one can use the domination $K^{i,U}\le K$, Lemma B.7(b) to get that $K^{i,U}(\{t\},dx)=0$ for any $t\in[0,T]\setminus\mathcal{G}_\varepsilon$. For P-a.s. ω, take arbitrary $(t,x)\in([0,T]\setminus\mathcal{G}_\varepsilon)\times\mathbb{R}$ and $\{(t_k,z_k)\}_{k\ge1}$, such that $\lim_{k\to\infty}(t_k,z_k)=(t,x)$. Then by Lemma B.7(b), we get that $\{\mathbf{1}(s<t_k)p_{t_k-s}(z_k-y)\}$ is uniformly integrable with respect to $K(ds,dy)$ and hence by domination it is also uniformly integrable with respect to $K^{i,U}(ds,dy)$. This gives continuity of the mapping

$$(r,x)\mapsto \int_0^r \int_{\mathbb{R}} p_{r-s}(x-y)K^{i,U}(ds,dy)$$

on $([0,T]\setminus\mathcal{G}_\varepsilon)\times\mathbb{R}$, and again by domination we may easily show that

$$r\mapsto \int_0^r \int_{\mathbb{R}} p_{r-s}(\cdot-y)K^{i,U}(ds,dy)\in C([0,T]\setminus\mathcal{G}_\varepsilon,C_{\mathrm{rap}}^+).$$

All together, this gives that the right-hand side of (B.55) belongs to $C([0,T]\setminus\mathcal{G}_\varepsilon,C_{\mathrm{rap}})$. Hence there is a version of U^i which is in $C([0,T]\setminus\mathcal{G}_\varepsilon,C_{\mathrm{rap}}^+)$ as well.

Note that, in fact, the above argument also easily implies that for any $t\in\mathcal{G}_\varepsilon$,

$$U^i(r,\cdot)\to U^i(t-,\cdot) \qquad \text{in } C_{\mathrm{rap}}, P\text{-a.s.} \tag{B.56}$$

as $r\uparrow t$, where

$$
\begin{aligned}
U^i(t-,x) &= \mathbf{1}(t>s_i)\int_{\mathbb{R}} p_{t-s_i}(x-y)J_\varepsilon^{x_i}(y)\,dy \\
&\quad + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)U(s,y)^{\gamma-1/2}U^i(s,y)^{1/2}W^{i,U}(ds,dy)
\end{aligned} \tag{B.57}
$$

$$- \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)(K^{i,U}(ds,dy) - \delta_t(ds)K^{i,U}(\{t\},dy))$$

for $x \in \mathbb{R}$. Indeed, for $(t,x) \in \mathcal{G}_\varepsilon \times \mathbb{R}$, take again arbitrary (t_k, z_k) such that $t_k \uparrow t$ and $z_k \to x$, as $k \to \infty$. Again by Lemma B.7(b), we get that $\{\mathbf{1}(s < t_k)p_{t_k-s}(z_k - y)\}$ is uniformly integrable with respect to $(K(ds,dy) - K(\{t\},dy))$; hence by domination it is also uniformly integrable with respect to $(K^{i,U}(ds,dy) - \delta_t(ds)K^{i,U}(\{t\},dy))$. This easily implies that $U^i(t_k,z_k) \to U^i(t-,x)$, where $U^i(t-,x)$ satisfies (B.57), and hence (B.56) follows.

Clearly, (B.56) implies that corresponding convergence also holds in $L^1(\mathbb{R})$, and hence to finish the proof of the lemma it is enough to show that for any $t \in \mathcal{G}_\varepsilon$,

$$U^i(r,\cdot) \to U^i(t,\cdot) \qquad \text{in } L^1(\mathbb{R}), P\text{-a.s.} \tag{B.58}$$

as $r \downarrow t$. Again, as in the proof of Lemma B.7(b), we will show it for $t = s_j \in \mathcal{G}_\varepsilon^{\mathrm{odd}}$ for some j. By (B.48), we get that

$$U^i_{s_j}(dx) = U^i_{s_j-}(dx) + \mathbf{1}(s_i = s_j)J^{x_i}_\varepsilon(x)\,dx - K^{i,U}(\{s_j\},dx). \tag{B.59}$$

Recall again that $K^{i,U}(\{s_j\},dx)$ is dominated by $K(\{s_j\},dx)$, which, in turn, by (B.35) is absolutely continuous with a density function in C^+_{rap}. Therefore $K^{i,U}(\{s_j\},dx)$ is also absolutely continuous with a density function $K^{i,U}(\{s_j\},x), x \in \mathbb{R}$, bounded by a function in C^+_{rap}. This together with (B.56), our assumptions on $J^{x_i}_\varepsilon$ and (B.59) implies that $U^i_{s_j}(dx)$ is absolutely continuous with bounded density function

$$U^i_{s_j}(\cdot) \in L^1(\mathbb{R}). \tag{B.60}$$

For any $\eta \in (0, \varepsilon/2)$, by combining (B.59), (B.56) (with $t = s_j$) and (B.55) (with $t = s_j + \eta$), we have

$$\begin{aligned} U^i(s_j+\eta,\cdot) &= S_\eta U^i(s_j,\cdot) \\ &\quad + \int_{s_j}^{s_j+\eta}\int_{\mathbb{R}} p_{s_j+\eta-s}(\cdot - y)U(s,y)^{\gamma-1/2}U^i(s,y)^{1/2}W^{i,U}(ds,dy) \\ &\quad - \int_{s_j}^{s_j+\eta}\int_{\mathbb{R}} p_{s_j+\eta-s}(\cdot - y)(K^{i,U}(ds,dy) - \delta_{s_j}(ds)K^{i,U}(\{s_j\},dy)) \end{aligned} \tag{B.61}$$

for $x \in \mathbb{R}$. As $\eta \downarrow 0$, the convergence to zero in C_{rap} of the second and the third terms on the right-hand side follows easily as in the last part of the proof of Lemma B.7(b). By (B.60), the first term on the right-hand side of (B.61) converges to $U^i(s_j,\cdot)$ in $L^1(\mathbb{R})$ and we are done. $\square$

Proof of Proposition B.2. Except for property (2.3), Proposition B.2 follows from Corollary B.4, and Lemmas B.8(a), B.9, B.10. For (2.3) we note that

$$U^i(t,x)V^j(t,x) \le U(t,x)V(t,x) = w^+(t,x)w^-(t,x) \equiv 0. \qquad \square$$

Proof of Proposition 2.1. As we mentioned in Remark B.1, since $T > 1$ can be chosen arbitrary large, it is sufficient to prove the theorem on the time interval $[0,T]$.

Clearly, by Proposition B.2 and the definition of $\bar{U}^i = U^i + \widetilde{U}^i, \bar{V}^i = V^i + \widetilde{V}^i$, we immediately get that

$$(\bar{U}^i, \bar{V}^i) \in (C([0,T] \setminus \mathcal{G}_\varepsilon, C^+_{\mathrm{rap}}) \cap D^\varepsilon([0,T], L^1(\mathbb{R})))^2, \qquad i \in \mathbb{N}_\varepsilon,$$

and satisfies (2.6) and (2.7). We saw in Section 2 that (2.5) and its analogue for (U^i, V^j) follow from the other properties. Then, by repeating the argument in the proof of Lemma B.10 and taking into account the absence of the terms $K^{i,U}, K^{i,V}$ at the right-hand side of the equations for $\bar{U}^i, \bar{V}^i$, we immediately get that, in fact, $(\bar{U}^i, \bar{V}^i) \in D^\varepsilon([0,T], C^+_{\mathrm{rap}})^2, i \in \mathbb{N}_\varepsilon$, and $\bar{U}^i_{s_i+\cdot} \in C([0, T - s_i], C^+_{\mathrm{rap}})$, $\bar{V}^i_{t_i+\cdot} \in C([t_i, T - t_i], C^+_{\mathrm{rap}}), i \in \mathbb{N}_\varepsilon$, and part (a) of the theorem follows. Part (b) follows from Lemma B.7(c). $\square$

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