HITTING PROPERTIES AND NON-UNIQUENESS FOR SDES DRIVEN BY STABLE PROCESSES

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Abstract. We study a class of self-similar jump type SDEs driven by Hölder continuous drift and noise coefficients. Using the Lamperti transformation for positive self-similar Markov processes we obtain a necessary and sufficient condition for almost sure extinction in finite time. We then show that in some cases pathwise uniqueness holds in a restricted sense, namely among solutions spending a Lebesgue-negligible amount of time at 0. A direct power transformation plays a key role.

1. Introduction and Results

In recent years there has been considerable interest in proving existence and especially uniqueness of solutions to stochastic differential equations (SDEs) driven by α-stable Lévy processes with Hölder continuous coefficients. In [13, 20, 12, 3, 16] pathwise uniqueness for equations with Hölder continuous noise coefficients was obtained in the spirit of the classical Yamada-Watanabe result for SDEs driven by Brownian motion. On the other hand when the noise is additive (i.e. the noise coefficient is constant) and it is the drift coefficient which is supposed to be Hölder continuous, Priola [21] extended the known results for SDEs driven by Brownian motion to SDEs driven by stable Lévy processes.

In the present work we want to focus on a family of SDEs interpolating between the two classes of problems described above since both the drift and the noise coefficients are chosen Hölder continuous. We study existence and uniqueness (or lack of uniqueness) of non-negative solutions $(Z_t)_{t\geq 0}$ to the stochastic differential equation of jump type

$$Z_t = Z_0 + \int_0^t Z_{s-}^{\beta}\, dL_s + \theta \int_0^t Z_s^{\eta}\, ds, \qquad t \geq 0. \tag{1.1}$$

Here $\beta, \eta \in [0,1), \theta \geq 0$ and $(L_t)_{t\geq 0}$ is a spectrally positive α-stable, $\alpha \in (1,2)$, Lévy process with Laplace exponent

$$\log \mathbb{E}[e^{-\lambda L_1}] = \lambda^{\alpha} = \int_0^{\infty} \left(e^{-\lambda x} - 1 + \lambda x\right) c_{\alpha}\, x^{-1-\alpha}\, dx, \qquad \lambda \geq 0,$$

with normalizing constant $c_{\alpha} = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}$. Observe in any case that the drift is non-locally Lipschitz precisely around a point where the noise coefficients is degenerate

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(i.e. equal to 0). Therefore, it is perhaps not so surprising that uniqueness might fail if solutions hit zero. One of our main results is that, in a certain regime, pathwise uniqueness indeed fails, and we can explicitly construct two different solutions.

If we choose η properly as a function of α and β, the solutions of (1.1) trapped in 0 will be self-similar. This, in fact, is a crucial ingredient for our analysis and we will henceforth assume

$$\theta \geq 0, \quad \alpha \in (1,2), \quad \beta \in [1-1/\alpha, 1), \quad \eta = 1-\alpha(1-\beta) \in [0,1). \tag{1.2}$$

The choice of parameters (1.2) might appear arbitrary but it covers important special cases:

- Solutions to (1.1) with $\theta = 0$ and $\beta = 1/\alpha$ are called continuous state branching processes with stable branching mechanism. If $\theta > 0$ and still $\beta = 1/\alpha$, the additional drift can be interpreted as a state-dependent immigration to the system and was studied for more general immigration mechanisms in Chapter 10 of Li [19].
- In the forthcoming article Berestycki et al. [5] the authors use a spatial version of the SDE (1.1) with $\beta = 1/\alpha$ and $\eta = 2-\alpha$ to study generalized Fleming-Viot superprocesses with mutation. Since the problem of existence and uniqueness of solutions to (1.1) seems to be of independent interest, it is studied here separately.
- For the boundary case $\beta = 1-1/\alpha$ the drift becomes constant since $\eta = 0$. Hence, for any $\theta \geq 0$ the pathwise uniqueness holds due to the results of Li and Mytnik [20]. It was shown in [20] that, for any $\beta \geq 1-1/\alpha$, pathwise uniqueness holds for non-negative solutions of the SDE similar to (1.1) but with Lipschitz conditions on drift coefficient b:

  $$Z_t = Z_0 + \int_0^t |Z_{s-}|^\beta \, dL_s + \int_0^t b(Z_s)\, ds, \qquad t \geq 0.$$

  For completeness we should mention that for this SDE driven by *symmetric* α-stable process, pathwise uniqueness holds among all solutions, for any $\beta \geq 1/\alpha$. Moreover the result for the symmetric case is sharp: there is a counter example of non-uniqueness if $\beta < 1/\alpha$ (see Bass [3] and Komatsu [16]).
- The second boundary case $\beta = 1$ is not covered by the range of parameters defined in (1.2). Nevertheless, it implies that $\eta = 1$ and for any parameter $\theta \geq 0$ the SDE (1.1) is a linear equation for which pathwise uniqueness is a simple consequence of the Lipschitz property of the coefficients.

Before stating the results let us fix some notation. We suppose that $(L_t)_{t\geq 0}$ is adapted to a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. A $(\mathcal{G}_t)_{t\geq 0}$-adapted stochastic process $(Z_t)_{t\geq 0}$ with almost surely *càdlàg* sample paths solving (1.1) a.s. is said to be a *solution to Equation* (1.1). If a solution is adapted to the augmented natural filtration of $(L_t)_{t\geq 0}$ then it is said to be a *strong solution to Equation* (1.1). We say that pathwise uniqueness holds for the SDE (1.1) if for any two solutions Z^1, Z^2 defined on Ω we have $\mathbb{P}(Z^1_t = Z^2_t, \, \forall\, t \geq 0) = 1$.

A first simple observation is that strong existence and pathwise uniqueness of non-negative solutions hold for the SDE (1.1) before

$$T_0 := \inf\{t \geq 0 : Z_t = 0\},$$

the first hitting time of 0; indeed, the coefficients are locally Lipschitz continuous on (ε, ∞) for all $\varepsilon > 0$. In order to understand why uniqueness might fail, we first explain when the event $\{T_0 < \infty\}$ has positive probability, since otherwise nothing needs to be proved. Using that solutions of Equation (1.1) are self-similar for the appropriate choice of β and η as in (1.2), Lamperti's transformation – which will be recalled below – can be applied to prove the following result:

Theorem 1.1. *Suppose α, β and η are chosen as in* (1.2) *and let $Z_0 > 0$. Then $T_0 < \infty$ almost surely if and only if $0 \leq \theta < \Gamma(\alpha)$.*

To see how uniqueness might fail when (re)starting at zero let us suppose that Z is the unique solution up to T_0. If $\beta > 1 - 1/\alpha$, the solution trapped at 0 $(\bar{Z}_t)_{t\geq 0} := (Z_{t\wedge T_0})_{t\geq 0}$ solves (1.1). Hence, the existence of a non-trivial solution after T_0 contradicts pathwise uniqueness in the classical sense; however in such cases we can study pathwise uniqueness in a weaker sense, namely in the class $\mathcal{S}$ of non-sticky solutions

$$\mathcal{S} := \left\{ (Z_t)_{t\geq 0} \,\middle|\, Z \geq 0 \text{ and } \int_0^\infty \mathbb{1}_{\{Z_t = 0\}}\, dt = 0 \quad \text{a.s.} \right\}$$

which in particular rules out the trapped solution $\bar{Z}$. Of course it is not clear *a priori* whether there is a solution $Z \in \mathcal{S}$. Both the drift and the noise are null when solutions hit zero so that existence of strong solutions leaving zero is non-trivial. The following theorem is one of the main results of the paper.

Theorem 1.2. *Suppose α, β and η are chosen as in* (1.2) *and let $Z_0 > 0$.*

A) *If $\theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, then pathwise uniqueness for the SDE* (1.1) *holds among solutions in $\mathcal{S}$. Moreover, a strong solution $Z \in \mathcal{S}$ exists.*

B) *If $\theta \in [0, \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}]$, then there exists a unique non-negative solution to the SDE* (1.1)*, which is moreover strong. The solution hits zero with probability one, and then stays at zero forever.*

Remark 1.3. In 2007, for a different equation Bass et al. have obtained a strong uniqueness result for solutions restricted to a class similar to $\mathcal{S}$.

To combine the two theorems notice that with the choice (1.2) of parameters α, β, η the inequality

$$\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < \Gamma(\alpha) \tag{1.3}$$

holds, see Lemma 2.5 below. Therefore, for $\beta \in (1 - 1/\alpha, 1)$ and $Z_0 > 0$, Theorems 1.1 and 1.2 define three regimes for the SDE (1.1) as θ varies:

- If $\theta \leq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ then the set of solutions of type $\mathcal{S}$ is empty.

- If $\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < \theta < \Gamma(\alpha)$ there is a unique, strong, non-trivial solution in $\mathcal{S}$ which hits zero in finite time almost surely. Since the trapped solution $\bar{Z}$ is a strong solution, in this regime we find a non-uniqueness phenomenon for solutions to the SDE (1.1).
- If $\theta \geq \Gamma(\alpha)$ there is a unique strong solution which never hits zero. The pathwise uniqueness is not restricted to $\mathcal{S}$ in this case, and holds among all non-negative solutions.

It is interesting to note that the three regimes can be equally obtained from the theory of positive self-similar Markov processes as we explain below in Section 4.

Let us next discuss the connection to the particular boundary case $\alpha = 2, \beta = 1/\alpha = 1/2$ (which is not covered by (1.2)). Equation (1.1) becomes

$$Z_t = Z_0 + \int_0^t \sqrt{2Z_s}\, dB_s + \theta\, t, \qquad t \geq 0,$$

for $\theta \geq 0$ and $Z_0 \geq 0$. We recognize in $(2Z_t, t \geq 0)$ a squared-Bessel process of dimension 2θ. The drift is constant, and pathwise uniqueness always holds due to the classical results of Yamada and Watanabe. Since $\Gamma(0) = +\infty$, the interesting regime B in Theorem 1.2 reduces to the case $\theta = 0$, where 0 is a trap for Z by pathwise uniqueness. Since $\Gamma(2) = 1$, the dichotomy of Theorem 1.1 corresponds to the fact that a Bessel process of dimension 2θ hits 0 in finite time with positive probability iff $\theta < 1$ (see for instance Chapter XI of [23]).

We now explain how the above results can be extended to more general equations where we replace the constant θ by a globally Lipschitz continuous function f. For the SDE

$$Z_t = Z_0 + \int_0^t Z_{s-}^{\beta}\, dL_s + \int_0^t Z_s^{\eta} f(Z_s)\, ds, \qquad t \geq 0, \tag{1.4}$$

we can prove the following generalizations of Theorems 1.1 and 1.2.

Theorem 1.4. *Suppose α, β and η are chosen as in (1.2) and let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a Lipschitz continuous function such that $x^\eta f(x)$ has at most linear growth as $x \to +\infty$. Then the following hold:*

A1) *If $Z_0 > 0$ and*

$$\sup_{r \geq 0} f(r) < \Gamma(\alpha), \qquad \forall\, r \geq 0, \tag{1.5}$$

then $T_0 < \infty$ almost surely.

A2) *If $Z_0 > 0$ and*

$$f(0) < \Gamma(\alpha), \tag{1.6}$$

then $\mathbb{P}_{Z_0}(T_0 < \infty) > 0$.

B) *If $Z_0 > 0$ and there exists $\delta > 0$*

$$f(r) \geq \Gamma(\alpha), \qquad \forall\, r \in [0, \delta], \tag{1.7}$$

then $T_0 = \infty$ almost surely.

If $f(\cdot) \geq \Gamma(\alpha)$ in a neighborhood of zero, then we can clearly use Theorem 1.4 to conclude that there is a unique strong solution to (1.4) which never hits zero. Again, the pathwise uniqueness is not restricted to $\mathcal{S}$ in this case, and holds for all non-negative solutions. Moreover, the assumptions on f could be relaxed, and it is enough to assume that f is Lipschitz on $(0, \infty)$ in this case. For example, pathwise uniqueness holds for any $f(x) = x^{-v}, x > 0$, with $v \in (0, \eta]$.

Theorem 1.5. *Suppose α, β and η are chosen as in* (1.2) *and let f be a Lipschitz continuous function on $[0, \infty)$ such that $x^\eta f(x)$ has at most linear growth as $x \to +\infty$ and let $Z_0 > 0$. Then the following holds:*

A) *If $f(0) > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, then pathwise uniqueness for the SDE* (1.4) *holds among solutions in $\mathcal{S}$. Moreover, a strong solution in $\mathcal{S}$ exists.*
B) *If $f(0) \leq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, then there exists a unique non-negative solution to the SDE* (1.4)*, which is moreover strong. If the solution hits zero, then it stays at zero forever.*

Organization of the Proofs. In Section 2 we use the theory of positive self-similar Markov processes to prove Theorems 1.1 and 1.4. The arguments for the proof of Theorem 1.5 are gathered in Section 3 and Theorem 1.2 follows from it as a particular case. Finally, in Section 4 we show how our results can be used to construct self-similar extensions of $(Z_{t\wedge T_0})_{t\geq 0}$ solving (1.1).

2. Self-similarity and the Proof of Theorems 1.1 and 1.4

A positive self-similar Markov process (pssMp) of index γ is a strong Markov family $(\mathbb{P}^x)_{x>0}$ with coordinate process denoted by Z in the Skorohod space of càdlàg functions satisfying

$$\text{the law of } (cZ_{c^{-1/\gamma}t})_{t\geq 0} \text{ under } \mathbb{P}^x \text{ is given by } \mathbb{P}^{cx} \tag{2.1}$$

for all $c > 0$. John Lamperti has shown in [18] that this property is equivalent to the existence of a Lévy process ξ such that, under $\mathbb{P}^x$, the process $(Z_{t\wedge T_0})_{t\geq 0}$ has the same law as $\left(x \exp\left(\xi_{\tau(tx^{-1/\gamma})}\right)\right)_{t\geq 0}$, where

$$\tau(t) := \inf\{s \geq 0 : A_s > t\} \qquad \text{and} \qquad A_t := \int_0^t \exp\left(\frac{1}{\gamma}\xi_s\right) ds.$$

Since this is all we need, we assume from now on that the Lévy process ξ is conservative, i.e. the lifetime is infinite. The proof of Theorem 1.1 is based on the equivalence

$$T_0 < \infty \quad \text{a.s. for all initial conditions } Z_0 > 0 \quad \Longleftrightarrow \quad \xi \text{ drifts to } -\infty \tag{2.2}$$

for pssMps which is due to Lamperti [18]. In order to connect the SDE (1.1) to these results we start with a simple lemma.

Lemma 2.1. *Suppose α, β and η are chosen as in* (1.2)*. Then, for any initial condition $x > 0$, the SDE (1.1) admits a unique non-negative solution absorbed at zero. The induced Markov family $(\mathbb{P}^x)_{x>0}$ is self-similar of index* $1/(1-\eta) \geq 1$.

Proof. Existence and pathwise uniqueness before hitting any level ϵ follows from the Lipschitz structure of the integrands in (ϵ,∞). Sending ϵ to zero this carries over to solutions up to T_0. To prove the self-similarity assertion, we abbreviate $\gamma = 1/(1-\eta)$ to obtain

$$\begin{aligned} cZ_{tc^{-1/\gamma}} &= cZ_0 + \int_0^{tc^{-1/\gamma}} cZ_{s-}^{\beta}\, dL_s + \theta \int_0^{tc^{-1/\gamma}} cZ_s^{1-1/\gamma} ds \\ &= cZ_0 + \int_0^t cZ^{\beta}_{(sc^{-1/\gamma})-} dL_{(c^{-1/\gamma}s)} + \theta \int_0^t c^{1-1/\gamma} Z^{1-1/\gamma}_{sc^{-1/\gamma}} ds \\ &= cZ_0 + \int_0^t \left(cZ_{(sc^{-1/\gamma})-}\right)^{\beta} dL_s^c + \theta \int_0^t \left(cZ_{sc^{-1/\gamma}}\right)^{1-1/\gamma} ds, \end{aligned}$$

where the Lévy process $L_t^c := c^{1/(\alpha\gamma)} L_{tc^{-1/\gamma}}$ has the same distribution as L, and we have used in particular that $c^{1-1/(\alpha\gamma)} = c^{\beta}$. The self-similarity now follows from well-posedness of the SDE before hitting zero. □

Next, we calculate the Lévy process ξ corresponding to solutions of the SDE (1.1) via Lamperti's transformation. For $\theta = 0$ and $\beta = 1/\alpha$, i.e. the stable CSBP without immigration, ξ can be recovered from Proposition 2 of Kyprianou and Pardo [17] combined with the generator calculations of Caballero and Chaumont [7].

Lemma 2.2. *Suppose that $\mathcal{M}$ is a Poisson point process on $(0,\infty)\times(0,\infty)$ with intensity measure $\mathcal{M}'(ds,dx) = ds \otimes c_\alpha e^x (e^x-1)^{-\alpha-1}\, dx$, then*

$$\xi_t := \left(\theta + \int_0^\infty (\log(1+x) - x)\, c_\alpha x^{-1-\alpha}\, dx\right) t + \int_0^t \int_0^\infty x\, (\mathcal{M} - \mathcal{M}')(ds,dx) \tag{2.3}$$

is the Lévy process corresponding under Lamperti's transformation to the pssMp $(\mathbb{P}^x)_{x>0}$ defined by the SDE (1.1).

Proof. First note that ξ can equivalently be written as

$$\begin{aligned} \xi_t = &\left(\theta + \int_0^\infty (\log(1+x) - x)\, c_\alpha x^{-1-\alpha}\, dx\right) t \\ &+ \int_0^t \int_0^\infty \log(1+x) (\mathcal{N} - \mathcal{N}')(ds,dx), \end{aligned} \tag{2.4}$$

where $\mathcal{N}$ is a Poisson point process on $(0,\infty)\times(0,\infty)$ with intensity measure $\mathcal{N}'(ds,dx) = ds \otimes c_\alpha x^{-1-\alpha}$. The equivalence follows for instance from Theorem

II.1.8 of Jacod and Shiryaev [15] and the compensator calculation (for any measurable function W with compact support in $(0,\infty)\times\mathbb{R}\backslash\{0\}$)

$$\begin{aligned}\int_0^t\int_0^\infty W(s,x)\mathcal{M}'(ds,dx) &= \int_0^t\int_0^\infty W(s,x)c_\alpha e^x(e^x-1)^{-1-\alpha}\,dx\\ &= \int_0^t\int_0^\infty W(s,\log(x+1))\,c_\alpha\,x^{-1-\alpha}\,dx\\ &= \int_0^t\int_0^\infty W(s,\log(x+1))\,\mathcal{N}'(ds,dx)\end{aligned}$$

so that the jump-measures of both Poissonian integrals have the same deterministic intensity. Itô's formula (see page 44 of Ikeda and Watanabe [14]) applied to (2.4) directly shows that $M_t:=\exp(\xi_t)$ satisfies

$$M_t = 1+\theta\int_0^t M_s\,ds+\int_0^t\int_0^\infty M_{s-}x\,(\mathcal{N}-\mathcal{N}')(ds,dx). \tag{2.5}$$

Recalling from the Lévy-Itô representation that

$$L_t := \int_0^t\int_0^\infty x\,(\mathcal{N}-\mathcal{N}')(ds,dx)$$

is a spectrally positive α-stable Lévy process with Laplace exponent λ^α and inserting in (2.5) shows that $\exp(\xi_t)$ solves

$$M_t = 1+\theta\int_0^t M_s\,ds+\int_0^t M_{s-}\,dL_s.$$

Next, we have to include the time-change: since $\gamma=1/(1-\eta)$, Lamperti's time-change becomes

$$\tau(t):=\inf\{s>0: A_s>t\},\quad A_t:=\int_0^t\lambda_s\,ds,\quad \lambda_s:=\exp\{(1-\eta)\xi_s\}.$$

If we set

$$\tilde{L}_t := \int_0^{\tau(t)}\lambda_{s-}^{1/\alpha}\,dL_s,\qquad t>0,$$

then we claim that $(\tilde{L}_t)_{t\geq 0}$ has the same law as $(L_t)_{t\geq 0}$. Indeed, let us denote by $\tilde{\mathcal{N}}$, respectively $\tilde{\mathcal{N}}'$, the image measure of $\mathcal{N}$, resp. $\mathcal{N}'$, under the map $(s,x)\mapsto(A_s,\lambda_{s-}^{1/\alpha}x)$. Then $\tilde{\mathcal{N}}$ is an optional random measure, whose compensator $\tilde{\mathcal{N}}'$ is equal to $\mathcal{N}'$, since using the change of variable $(A_s,\lambda_{s-}^{1/\alpha}x)=(r,y)$, we find

$$\int_0^\infty\int_0^\infty W(A_s,\lambda_{s-}^{1/\alpha}x)\,dsc_\alpha x^{-1-\alpha}\,dx = \int_0^\infty\int_0^\infty W(r,y)\,\lambda_{\tau(r)}^{-1-\frac{1}{\alpha}+\frac{1}{\alpha}+1}\,dr\,c_\alpha y^{-1-\alpha}dy.$$

By [15, Theorem II.1.8], $\tilde{\mathcal{N}}$ and $\mathcal{N}$ have the same law. Therefore, since

$$\int_0^t \int_0^\infty x\,(\tilde{\mathcal{N}} - \tilde{\mathcal{N}}')(ds,dx) = \int_0^\infty \int_0^\infty \mathbb{1}_{(A_s \le t)}\, \lambda_{s-}^{1/\alpha}\, x\,(\mathcal{N} - \mathcal{N}')(ds,dx)$$
$$= \int_0^{\tau(t)} \lambda_{s-}^{1/\alpha}\, dL_s = \tilde{L}_t,$$

the claim is proved. Plugging-in, we obtain

$$\begin{aligned} M_{\tau(t)} &= 1 + \theta \int_0^{\tau(t)} M_s\, ds + \int_0^{\tau(t)} M_{s-} dL_s \\ &= 1 + \theta \int_0^t M_{\tau(u)}\, \dot{\tau}(u)\, du + \int_0^t M_{\tau(u)-} e^{-(1-\beta)\xi_{\tau(u)-}} d\tilde{L}_u \\ &= 1 + \theta \int_0^t e^{\xi_{\tau(u)}(1-(1-\eta))}\, du + \int_0^t e^{\xi_{\tau(u)-}(1-(1-\beta))} d\tilde{L}_u \\ &= 1 + \theta \int_0^t M_{\tau(u)}^{\eta}\, du + \int_0^t M_{\tau(u)-}^{\beta} d\tilde{L}_u. \end{aligned}$$

Hence, $(M_{\tau(t)})_{t \le T_0}$ is a weak solution to the SDE (1.1) until first hitting zero. Uniqueness of solutions then implies that ξ is the Lamperti transformed Lévy process corresponding to the solution of the SDE (1.1). □

Corollary 2.3. *Suppose that ξ is the Lamperti transformed Lévy process corresponding to the pssMp $(\mathbb{P}^x)_{x>0}$ induced by the solutions to the SDE (1.1), then*

$$\mathbb{E}[\exp(\lambda \xi_1)] = \exp\left(\lambda \left(\theta - \frac{\Gamma(\alpha - \lambda)}{\Gamma(1-\lambda)}\right)\right) \tag{2.6}$$

for $\lambda \in [0,1)$.

Proof. All we need to do is to apply the exponential formula (see Theorem 25.17 of Sato [26])) to the Lévy process ξ represented as in (2.4)

$$\begin{aligned} \mathbb{E}[\exp(\lambda \xi_1)] &= \exp\Bigg(\lambda\left(\theta + \int_0^\infty (\log(1+x) - x)\, c_\alpha x^{-1-\alpha}\, dx\right) \\ &\qquad + \int_0^\infty \big((1+x)^\lambda - 1 - \lambda \log(1+x)\big) c_\alpha x^{-1-\alpha}\, dx\Bigg) \\ &= \exp\left(\lambda\theta + \int_0^\infty \big((1+x)^\lambda - 1 - \lambda x\big) c_\alpha x^{-1-\alpha}\, dx\right). \end{aligned}$$

To calculate the inner integral we use twice partial integration to obtain

$$\int_0^\infty \big((1+x)^\lambda - 1 - \lambda x\big) c_\alpha x^{-1-\alpha}\, dx = \frac{\lambda(\lambda-1)}{\alpha(\alpha-1)} c_\alpha \int_1^\infty x^{\lambda-2}(x-1)^{-\alpha+1}\, dx. \tag{2.7}$$

Substituting x by $1/y$ and recalling that $c_\alpha = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}$ then yields

$$\frac{\lambda(\lambda-1)}{\Gamma(2-\alpha)}\int_0^1 \frac{1}{y^{\lambda-2}}\Big(\frac{1}{y}-1\Big)^{-\alpha+1}\frac{1}{y^2}\,dy = \frac{\lambda(\lambda-1)}{\Gamma(2-\alpha)}\int_0^1 x^{\alpha-\lambda-1}(1-x)^{1-\alpha}\,dx. \quad (2.8)$$

The integral can be reformulated via Beta-functions to obtain equality with

$$\frac{\lambda(\lambda-1)}{\Gamma(2-\alpha)}B(\alpha-\lambda,2-\alpha) = \frac{\lambda(\lambda-1)}{\Gamma(2-\alpha)}\frac{\Gamma(\alpha-\lambda)\Gamma(2-\alpha)}{\Gamma(2-\lambda)} = -\lambda\frac{\Gamma(\alpha-\lambda)}{\Gamma(1-\lambda)}, \quad (2.9)$$

where for the first equality we used $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ and for the second $\Gamma(x+1)=x\Gamma(x)$. □

Corollary 2.4. *Let $\beta \in [1-1/\alpha,1)$ and suppose that ξ is the Lamperti transformed Lévy process corresponding to the pssMp $(\mathbb{P}^x)_{x>0}$ induced by the solutions to the SDE (1.1), then*

- i) *ξ drifts to $-\infty$ if and only if $\theta<\Gamma(\alpha)$,*
- ii) *there is $0<a<1-\eta$ such that $\mathbb{E}[e^{a\xi_1}]>1$ if and only if $\theta>\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$.*

Proof. Let us first recall that, by Hölder's inequality, the Laplace exponent $\psi(\lambda) := \log\mathbb{E}[e^{\lambda\xi_1}]$ is convex whenever it is well-defined. Furthermore, it satisfies $\psi(0)=0$ and $\psi'(0+)=\mathbb{E}[\xi_1]$.
i) To verify the claim it suffices to check for which values θ the mean $\mathbb{E}[\xi_1]$ is strictly negative which is equivalent to finding $\lambda>0$ such that $\psi(\lambda)<0$. By our explicit calculation we have $\psi(\lambda)=\lambda\left(\theta-\frac{\Gamma(\alpha-\lambda)}{\Gamma(1-\lambda)}\right)$, $\lambda\in[0,1)$, so that ξ drifts to $-\infty$ if and only if there is $\lambda>0$ such that $\theta<\frac{\Gamma(\alpha-\lambda)}{\Gamma(1-\lambda)}$. Since the Gamma-function is continuous on $(0,\infty)$ and $\Gamma(1)=1$ and $\lambda\mapsto\frac{\Gamma(\alpha-\lambda)}{\Gamma(1-\lambda)}$ is increasing, this is possible if and only if $\theta<\Gamma(\alpha)$.
ii) Let us first assume $\eta>0$. As the formula for the Laplace exponent is well-defined for $\lambda=1-\eta$, the left-hand side of the claim is equivalent to

$$\theta > \frac{\Gamma(\alpha-(1-\eta))}{\Gamma(1-(1-\eta))} = \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$$

due to the convexity of ψ. Similarly in the case of $\eta=0$, we extend continuously the Laplace exponent to $\lambda=1$, and by taking $\Gamma(0)=\infty$ (this we assume everywhere throughout the paper) we see that the left-hand side of the claim is equivalent to $\theta>0$ in this case. □

Next, we connect the regimes of the previous corollary, i.e. we verify (1.3) above:

Lemma 2.5. *Suppose $\alpha\in(1,2)$ and $\beta\in[1-1/\alpha,1)$, then*

$$\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < \Gamma(\alpha).$$

Proof. Let $p,q,m,n>0$ with $(p-m)>0$ and $(q-n)<0$; then we claim that

$$\Gamma(p+n)\,\Gamma(q+m) > \Gamma(p+q)\,\Gamma(m+n). \quad (2.10)$$

Let us first show that this implies the claim of the lemma. By the constraint on β, we can find n such that $\alpha(1-\beta) < n < 1$; since $\alpha > 1$ we can set

$$p := \alpha - n > 0, \quad q := n - \alpha(1-\beta) > 0, \quad m := 1 - n > 0,$$

and we obtain

$$p + n = \alpha, \quad q + m = 1 - \alpha + \alpha\beta = \eta, \quad p + q = \alpha\beta, \quad m + n = 1.$$

Moreover $p - m = \alpha - 1 > 0$ and $q - n = -\alpha(1-\beta) < 0$, so that the desired result is obtained.
Let us now prove (2.10). Define the maps $f, g : (0,1) \to [0,+\infty)$ by

$$f(x) := x^{p-m}, \qquad g(x) := (1-x)^{q-n}.$$

Let us consider a $B(m,n)$-variable X, with density

$$\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\, x^{m-1}(1-x)^{n-1}\, \mathbb{1}_{\{x\in(0,1)\}}\, dx.$$

As $(p-m) > 0$ and $(q-n) < 0$, the maps f and g are monotone increasing on $[0,1]$. Therefore, considering (X,Y) i.i.d. we obtain

$$0 < \mathbb{E}((f(X)-f(Y))(g(X)-g(Y))) = 2(\mathbb{E}(f(X)g(X)) - \mathbb{E}(f(X))\,\mathbb{E}(g(X)))$$

i.e.

$$\mathbb{E}(f(X)g(X)) > \mathbb{E}(f(X))\,\mathbb{E}(g(X)).$$

But this corresponds to

$$\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\,\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} > \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\,\frac{\Gamma(p)\Gamma(n)}{\Gamma(p+n)}\,\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\,\frac{\Gamma(m)\Gamma(q)}{\Gamma(m+q)}$$

and after cancellation we get (2.10). □

With the preparation finished, our first theorem can be proved.

Proof of Theorem 1.1. In Lemma 2.1 we proved that solutions to the SDE (1.1) absorbed at zero form a pssMps $(\mathbb{P}^x)_{x>0}$. The corresponding Lévy process has been characterized in Lemma 2.2. Combining Lamperti's Equivalence (2.2) with part i) of Corollary 2.4 the claim follows. □

With an additional comparison argument we can deduce Theorem 1.4.

Proof of Theorem 1.4. Observe that the condition that $x \mapsto x^\eta f(x)$ has at most linear growth as $x \to +\infty$ ensures that there is no explosion. Theorem 1.4 follows from Theorem 1.1 by a simple comparison argument:

A1) Let

$$\theta \in (\sup_{x>0} f(x), \Gamma(\alpha)). \tag{2.11}$$

Consider solutions Z' to (1.4) and Z to (1.1), both started from $Z_0 = Z'_0 = x > 0$, constructed on the same probability space and driven by the same stable process L. Note that, due to our Lipschitz assumptions, Z' and Z are unique strong solutions to the corresponding equations until the first time they hit zero (Theorem 6.2.3 of [1], which gives uniqueness for the globally Lipschitz case, can be easily adapted to this situation). Now we will use this and the fact the noise coefficient is monotone increasing to show that $Z_s \geq Z'_s$ until the first time one of them hits zero. Indeed, for any $\epsilon \in [0,1[$, let us define

$$\widetilde{T}_\epsilon = \inf\{t \geq 0 : \ \min(Z'_t, Z_t) \leq \epsilon\}.$$

Note that since our driving process L has only nonnegative jumps, we immediately get that $\min(Z'_{\widetilde{T}_\epsilon}, Z_{\widetilde{T}_\epsilon}) = \epsilon$. Now let us define

$$\zeta(t) = Z'_t - Z_t, \ \ t \in [0, \widetilde{T}_\epsilon].$$

Then by following the proof of Theorem 5.5 in [13] we get for any $\epsilon \in]0,1[$,

$$\begin{aligned} &\mathbb{E}\left[\zeta(t \wedge \widetilde{T}_\epsilon \wedge \tau_m)^+\right] \qquad (2.12)\\ &\leq \ \limsup_{k\to\infty} \mathbb{E}\left[\int_0^{t\wedge\widetilde{T}_\epsilon\wedge\tau_m} \phi'_k(\zeta(s-))\left((Z'_{s-})^\eta f(Z'_{s-}) - (Z_{s-})^\eta\theta\right) \mathbf{1}_{\zeta(s-)>0} ds\right], \end{aligned}$$

for some sequence of non-negative functions ϕ'_k bounded by 1. Then we have

$$\begin{aligned} &\mathbb{E}\left[\zeta(t \wedge \widetilde{T}_\epsilon \wedge \tau_m)^+\right] \qquad (2.13)\\ &\leq \ \limsup_{k\to\infty} \mathbb{E}\left[\int_0^{t\wedge\widetilde{T}_\epsilon\wedge\tau_m} \phi'_k(\zeta(s-))\left((Z'_{s-})^\eta f(Z'_{s-}) - (Z_{s-})^\eta\theta\right) \mathbf{1}_{\zeta(s-)>0}\, ds\right]\\ &= \ \limsup_{k\to\infty} \mathbb{E}\Bigg[\int_0^{t\wedge\widetilde{T}_\epsilon\wedge\tau_m} \phi'_k(\zeta(s-))\Big\{\left((Z'_{s-})^\eta - (Z_{s-})^\eta\right)^+ f(Z'_{s-})\\ &\qquad\qquad + (Z_{s-})^\eta\left(f(Z'_{s-}) - \theta\right)\Big\}\mathbf{1}_{\zeta(s-)>0}\, ds\Bigg]\\ &\leq \ \mathbb{E}\left[\int_0^{t\wedge\widetilde{T}_\epsilon\wedge\tau_m} \epsilon^{\eta-1}\left(Z'_{s-} - Z_{s-}\right)^+ f(Z'_{s-})\, ds\right], \end{aligned}$$

where the last inequality follows by (2.11), boundedness of ϕ'_k by 1 and easy bound on $\left((Z'_{s-})^\eta - (Z_{s-})^\eta\right)^+$, for $s \leq \widetilde{T}_\epsilon$. Then we immediately get

$$\mathbb{E}\left[\zeta(t \wedge \widetilde{T}_\epsilon \wedge \tau_m)^+\right] \ \leq \ \int_0^t \epsilon^{\eta-1}\theta E\left[\zeta(s \wedge \widetilde{T}_\epsilon \wedge \tau_m)\right] ds, \qquad (2.14)$$

and by Gronwall's inequality we get

$$\mathbb{E}\left[\zeta(t \wedge \widetilde{T}_\epsilon \wedge \tau_m)^+\right] = 0, \qquad (2.15)$$

for any $\epsilon \in]0,1[$, $m > 0$, $t \geq 0 > 0$. Since the above equality holds for all $\epsilon \in]0,1[$ and $\tau_m \to \infty$, as $m \to \infty$, we immediately get that

$$\zeta(t)^+ = 0, \forall t \leq \widetilde{T}_0\,,$$

and hence the comparison result

$$Z_t \geq Z'_t\,, \ \ \forall t \leq \widetilde{T}_0\,,$$

follows. Since, by Theorem 1.1, Z hits zero with probability one, from the above comparison, we immediately get that Z' hits zero with probability one, as well.

A2) To adapt the above arguments we only need to show that solutions enter a neighborhood of zero with positive probability. Indeed, suppose for any $\epsilon \in]0, Z_0[$, such that $f(y) < \Gamma(\alpha)$ for all $y \in [0, 2\epsilon]$, and for Z' being the solution of (1.4), then the following holds:

$$\mathbb{P}_{Z'_0}(T'_\epsilon < \infty) > 0, \tag{2.16}$$

where

$$T'_\epsilon = \inf\{t \geq 0 : \; Z'_t \leq \epsilon\}.$$

Let

$$\theta \in (\sup_{x\in[0,2\epsilon]} f(x), \Gamma(\alpha)). \tag{2.17}$$

Now let Z be a solution to (1.1), with θ as in (2.17), started at time T'_ϵ from $Z_{T'_\epsilon} = Z'_{T'_\epsilon} = \epsilon$, constructed on the same probability space as Z' and driven by the same stable process L (after the stopping time T'_ϵ). Fix

$$T''_\epsilon = \inf\{t \geq T'_\epsilon : \; Z_t \geq 2\epsilon\}.$$

It is clear from the argument in the proof of (A1) and by our construction, that

$$Z'_t \leq Z_t, \;\; \forall t \in [T'_\epsilon, \min\{T''_\epsilon, T_0\}], \tag{2.18}$$

where

$$T_0 = \inf\{t \geq T'_\epsilon : \; \min(Z'_t, Z_t) \leq 0\}.$$

Since Z hits zero with probability one, we immediately get that

$$\mathbb{P}(T_0 < T''_\epsilon) > 0.$$

This, the domination (2.18), and (2.16) imply claim in (A2). To finish the proof of (A2), we have to show (2.16). Let $\epsilon \in]0, Z_0[$ such that $f(y) < \Gamma(\alpha)$ for all $y \in [0, 2\epsilon]$. We define

$$\tau_t := \inf\left\{u > 0 : \int_0^u Z_s^{\alpha\beta}\, ds > t\right\}, \qquad t \geq 0,$$

and the process $Q_t := Z_{\tau_t} \mathbb{1}_{(\tau_t < +\infty)}$. We also define $T^Q_\epsilon := \inf\{t > 0 : Q_t \leq \epsilon\}$. An application of the Itô formula shows that $(Q_{t\wedge T^Q_\epsilon}, t \geq 0)$ has the same law as $(R_{t\wedge T^R_\epsilon}, t \geq 0)$ where

$$R_t = Z_0 + L_t + \int_0^t (R_s \vee \epsilon)^{\eta-\alpha\beta} f(R_s)\, ds$$

and $T^R_\epsilon := \inf\{t > 0 : R_t \leq \epsilon\}$. On the other hand, setting $K := \sup_{u\in[0,2Z_0]} f(u)$ and $U := \inf\{t > 0 : R_t > 2Z_0\}$, by comparison we have a.s. $R_{t\wedge U} \leq Y_{t\wedge U}$ for all $t \geq 0$, where

$$Y_t := Z_0 + L_t + t(2Z_0)^{\eta-\alpha\beta} K, \qquad t \geq 0.$$

Therefore, we are reduced to prove that with positive probability Y visits $[0, \epsilon]$ before $[2Z_0, +\infty[$.

Let $\mathbb{P}_z^Y$ be the law of $(Y_t, t \geq 0)$, where $Y_t := z + L_t + Ct$. Fix $0 < \epsilon < z < M$ and set

$$T_\epsilon := \inf\{t > 0 : Y_t \leq \epsilon\}, \qquad S_M := \inf\{t > 0 : Y_t \geq M\}.$$

We want to prove that $\mathbb{P}_z^Y(T_\epsilon < S_M) > 0$. Suppose that $\mathbb{P}_z^Y(T_\epsilon < S_M) = 0$; we are going to show that this implies $\mathbb{P}_z^Y(T_\epsilon < +\infty) = 0$; this is clearly absurd, since it implies that $\mathbb{P}_z^Y(Y_t < \epsilon) = 0$, while the law of L_t has full support in $\mathbb{R}$ for all $t > 0$.

We are going to show that, if $\mathbb{P}_z^Y(T_\epsilon < S_M) = 0$, then Y is going to spend an infinite amount of time oscillating between z and M before ever reaching ϵ. We set

$$S^{(1)} := S_M, \qquad U^{(1)} := \inf\{t > S^{(1)} : Y_t \leq z\},$$

$$S^{(k+1)} := \inf\{t > U^{(k)} : Y_t \geq M\}, \qquad U^{(k+1)} := \inf\{t > S^{(k)} : Y_t \leq z\},$$

with the usual convention $\inf \emptyset := +\infty$. Then on the event $\{U^{(k)} \leq t\}$, before time t the process Y has visited first $[M, +\infty[$ and then $[0, z]$ at least k times.

Let us prove by recurrence on k that $\mathbb{P}_z^Y(T_\epsilon < S^{(k)}) = 0$. For $k = 1$ this is the assumption made above that $\mathbb{P}_z^Y(T_\epsilon < S_M) = 0$. For any $k \geq 1$

$$\mathbb{P}_z^Y(T_\epsilon < S^{(k+1)}) = \mathbb{P}_z^Y(T_\epsilon < S^{(k)}) + \mathbb{P}_z^Y(S^{(k)} \leq T_\epsilon < S^{(k+1)}).$$

By the recurrence assumption, $\mathbb{P}_z^Y(T_\epsilon < S^{(k)}) = 0$. On the other hand, $\mathbb{P}_z^Y(S^{(k)} \leq T_\epsilon < U^{(k)}) = 0$, since $\epsilon < z < M$ and L has positive jumps only. Therefore, by the strong Markov property,

$$\mathbb{P}_z^Y(T_\epsilon < S^{(k+1)}) = \mathbb{P}_z^Y(U^{(k)} \leq T_\epsilon < S^{(k+1)}) = \mathbb{E}_z(\mathbb{1}_{\{U^{(k)} < +\infty\}} \, \mathbb{P}^Y_{Y_{U^{(k)}}}(T_\epsilon < S_M)).$$

Note that $\mathbb{P}_z^Y(Y_{U^{(k)}} = z) = 1$, since Y has no negative jumps. Therefore

$$\mathbb{P}_z^Y(T_\epsilon < S^{(k+1)}) = \mathbb{P}_z^Y(U^{(k)} < +\infty) \, \mathbb{P}_z^Y(T_\epsilon < S_M) = 0.$$

We obtain $\mathbb{P}_z^Y(T_\epsilon < S^{(k+1)}, \ \forall\, k \in \mathbb{N}) = 0$. By the strong Markov property, $(S^{(k+1)} - S^{(k)})_{k \geq 1}$ is i.i.d. with $S^{(k+1)} - S^{(k)} > 0$ a.s. by right-continuity of $(Y_t)_t$. Then

$$\lim_{k \to \infty} S^{(k+1)} \geq \lim_{k \to \infty} \sum_{i=1}^{k} \left(S^{(i+1)} - S^{(k)} \right) = +\infty$$

a.s. and we obtain that

$$\{T_\epsilon < +\infty\} \subset \cup_{k \geq 1} \{T_\epsilon < S^{(k+1)}\} \qquad \text{a.s.}$$

and therefore $\mathbb{P}_z^Y(T_\epsilon < +\infty) = 0$.

B) Again, an easy coupling argument in a neighborhood of 0 allows to conclude this part. □

In fact, we calculated in Corollary 2.4 more than we needed for the proof of Theorem 1.1 since part ii) was not used. The equivalence will be used later in Section 4.

3. Solutions after T_0 and the Proof of Theorems 1.2 and 1.5

The proof of Theorem 1.5 is based on the simple power transformation $z \mapsto z^{1-\eta}$ which turns the original drift $x \mapsto x^\eta f(x)$ into a globally Lipschitz continuous drift.

Lemma 3.1. *Suppose that α, β, η and f are as in Theorem 1.5 and suppose that $(Z_t)_{t\geq 0}$ is a non-negative (strong) solution to the SDE (1.1) started at $Z_0 > 0$. Then $(Z_t^{1-\eta})_{t\geq 0}$ is a non-negative (strong) solution to*

$$\begin{aligned} V_t = Z_0^{1-\eta} + (1-\eta)\left(\int_0^t \left(f(V_s^{1/(1-\eta)}) - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right)\mathbb{1}_{\{V_s\neq 0\}}\,ds\right) \\ + \int_0^t\int_0^\infty \left(\left(V_{s-}^{\frac{1}{(1-\eta)}} + V_{s-}^{\frac{\beta}{1-\eta}}\,x\right)^{1-\eta} - V_{s-}\right)(\mathcal{N}-\mathcal{N}')(ds,dx), \quad t\geq 0, \end{aligned} \tag{3.1}$$

where $\mathcal{N}$ is a Poisson point process on $(0,\infty)\times(0,\infty)$ with intensity measure $\mathcal{N}'(ds,dx) = ds \otimes c_\alpha x^{-1-\alpha}$.

Proof. Let us first rewrite the SDE (1.1) via the Lévy-Itô representation in the form

$$Z_t = Z_0 + \int_0^t Z_s^\eta f(Z_s)\,ds + \int_0^t\int_0^\infty Z_{s-}^\beta x\,(\mathcal{N}-\mathcal{N}')(ds,dx),$$

where $\mathcal{N}$ is the jump-measure of L which has intensity $\mathcal{N}'(ds,dx) = ds\otimes c_\alpha x^{-1-\alpha}\,dx$. We cannot directly apply Itô's formula with $F(z) = z^{1-\eta}$ since F is not smooth at the boundary of $[0,\infty)$ and cannot be extended to a concave function on $\mathbb{R}$. To surround this difficulty let us define $F_\epsilon(z) = (z+\epsilon)^{1-\eta}$, which is smooth for $z\in[0,\infty)$, and

$$G(z,x,\epsilon) = F_\epsilon(z + z^\beta\,x) - F_\epsilon(z) - F'_\epsilon(z)z^\beta x, \qquad z,x,\epsilon \geq 0.$$

Itô's formula then yields the almost sure identity

$$\begin{aligned} (Z_t+\epsilon)^{1-\eta} - (Z_0+\epsilon)^{1-\eta} = \ &(1-\eta)\int_0^t (Z_s+\epsilon)^{-\eta}\,Z_s^\eta\,f(Z_s)\,ds \\ + \int_0^t\int_0^\infty &\left((Z_{s-}+\epsilon+Z_{s-}^\beta\,x)^{1-\eta} - (Z_{s-}+\epsilon)^{1-\eta}\right)(\mathcal{N}-\mathcal{N}')(ds,dx) \\ + \int_0^t\int_0^\infty &G(Z_s,x,\epsilon)\,c_\alpha\,x^{-1-\alpha}\,dx =: I_t^\epsilon + II_t^\epsilon + III_t^\epsilon. \end{aligned}$$

In order to finish the proof we let ϵ tend to zero and show that the summands converge almost surely along a subsequence. It follows readily from dominated convergence that

$$\lim_{\epsilon\to 0} I_t^\epsilon = (1-\eta)\int_0^t f(Z_s)\mathbb{1}_{\{Z_s>0\}}\,ds$$

almost surely. Next, for III_t^ϵ we make the change of the variables $y = x\frac{Z_s^\beta}{Z_s+\epsilon}$ to get

$$III_t^\epsilon = c_\alpha\int_0^t\left(\frac{Z_s}{Z_s+\epsilon}\right)^{\alpha\beta} ds\int_0^\infty \left((1+y)^{1-\eta} - 1 - (1-\eta)y\right)y^{-1-\alpha}dy.$$

Using (2.7)-(2.9) with $\lambda = 1-\eta$, one obtains

$$III_t^\epsilon = -(1-\eta)\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\int_0^t \left(\frac{Z_s}{Z_s+\epsilon}\right)^{\alpha\beta} ds.$$

Now, we can apply the dominated convergence theorem to obtain almost surely

$$\lim_{\epsilon\downarrow 0} III_t^\epsilon = -(1-\eta)\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\int_0^t \mathbb{1}_{\{Z_s>0\}}\, ds.$$

Next, we have to deal with the term II_t^ϵ for which we first show L^p-convergence for some $p\in(\alpha,2)$. Let us abbreviate

$$H(z,x,\epsilon) := F_\epsilon(z+z^\beta x) - F_\epsilon(z) \geq 0$$

satisfying

$$\frac{d}{d\epsilon}H(z,x,\epsilon) = F'_\epsilon(z+z^\beta x) - F'_\epsilon(z) \leq 0.$$

Since $1-\eta = \alpha(1-\beta) < 1$, we can fix $p \in (\alpha, \frac{1}{1-\beta}\wedge 2)$. Applying Burkholder-Davis-Gundy inequality (see e.g. [10, Theorem VII.92]) we obtain

$$\begin{aligned}
&\mathbb{E}\Bigg[\bigg(\int_0^t\int_0^\infty \Big((Z_{s-}+Z_{s-}^\beta x)^{1-\eta} - Z_{s-}^{1-\eta}\Big)(\mathcal{N}-\mathcal{N}')(ds,dx) \\
&\qquad - \int_0^t\int_0^\infty \Big((Z_{s-}+\epsilon+Z_{s-}^\beta x)^{1-\eta} - (Z_{s-}+\epsilon)^{1-\eta}\Big)(\mathcal{N}-\mathcal{N}')(ds,dx)\bigg)^p\Bigg] \\
&\leq c_p\mathbb{E}\Bigg[\bigg(\int_0^t\int_0^\infty \big(H(Z_{s-},x,0)-H(Z_{s-},x,\epsilon)\big)^2\mathcal{N}(ds,dx)\bigg)^{p/2}\Bigg] \qquad (3.2)\\
&\leq c_p\mathbb{E}\Bigg[\int_0^t\int_0^\infty \big(H(Z_s,x,0)-H(Z_s,x,\epsilon)\big)^p\mathcal{N}(ds,dx)\Bigg] \\
&= c_p\mathbb{E}\Bigg[\int_0^t\int_0^\infty \big(H(Z_s,x,0)-H(Z_s,x,\epsilon)\big)^p\, c_\alpha\, x^{-1-\alpha}dx\, ds\Bigg],
\end{aligned}$$

where $c_p > 0$ is a constant coming from the Burkholder-Davis-Gundy inequality. Since H is positive and pointwise decreasing in ϵ, to show that the righthand side of (3.2) converges to zero, by monotone convergence theorem, it is enough to show the boundedness of

$$\mathbb{E}\Bigg[\int_0^t\int_0^\infty H(Z_s,x,0)^p\, c_\alpha\, x^{-1-\alpha}dx\, ds\Bigg]. \qquad (3.3)$$

To this end, make the change of variable $x = Z_s^{1-\beta} y$ (note that the integrand is zero whenever $Z_s = 0$) to obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^t \int_0^\infty H(Z_s, x, 0)^p ds\, c_\alpha\, x^{-1-\alpha}\, dx\right]\\&= \mathbb{E}\left[\int_0^t \int_0^\infty Z_s^{(p-1)(1-\eta)}((1+y)^{1-\eta} - 1)^p\, ds\, c_\alpha\, y^{-1-\alpha} dy\right].\end{aligned}$$

To bound the righthand side we use two bounds for the integrand. First, applying the Hölder property, gives

$$((1+y)^{1-\eta} - 1^{1-\eta})^p \le y^{p(1-\eta)}$$

and, secondly, we use the mean-value theorem with some $\zeta > 0$ to obtain

$$((1+y)^{1-\eta} - 1^{1-\eta})^p = (1-\eta)^p((1+\zeta)^{-\eta} y)^p \le (1-\eta)^p y^p.$$

Plugging-in and using Fubini's theorem, we derive the upper bound

$$\begin{aligned}&\mathbb{E}\left[\int_0^t \int_0^\infty H(Z_s, x, 0)^p\, ds\, c_\alpha\, x^{-1-\alpha} dx\right]\\&\le c_p \int_0^t \mathbb{E}\left[Z_s^{(p-1)(1-\eta)}\right]\, ds\, \int_0^\infty \min(y^p, y^{p(1-\eta)})\, c_\alpha\, y^{-1-\alpha}\, dy.\end{aligned} \tag{3.4}$$

For the latter integral we estimate

$$\begin{aligned}\int_0^\infty \min(y^p, y^{p(1-\eta)})\, c_\alpha\, y^{-1-\alpha}\, dy &\le c_\alpha \int_0^1 y^{p-1-\alpha}\, dy + c_\alpha \int_1^\infty y^{p(1-\eta)-1-\alpha}\, dy\\&= c_\alpha \int_0^1 y^{p-1-\alpha}\, dy + c_\alpha \int_1^\infty y^{p\alpha(1-\beta)-1-\alpha}\, dy\end{aligned} \tag{3.5}$$

which is finite since $p \in (\alpha, (1-\beta)^{-1} \wedge 2)$. Defining $\tau_m = \inf\{t \ge 0 : Z_t > m\}$, which tends to infinity as $m \to +\infty$ almost surely by Lemma 2.3 of Fu and Li [13], the defining equation (1.1) yields, since $x \mapsto x^\eta f(x)$ grows at most linearly,

$$\begin{aligned}\mathbb{E}[Z_{t\wedge\tau_m}] = Z_0 + \mathbb{E}\left[\int_0^{t\wedge\tau_m} Z_s^\eta f(Z_s)\, ds\right] &\le Z_0 + C\int_0^t (\mathbb{E}[Z_{s\wedge\tau_m}] + 1)\, ds\\&\le Z_0 + C\int_0^t \mathbb{E}[Z_{s\wedge\tau_m}]\, ds + Ct.\end{aligned}$$

Hence, by the Gronwall inequality and Fatou's lemma,

$$\begin{aligned}\mathbb{E}\big[Z_t^{(p-1)(1-\eta)}\big] \le \lim_{m\to\infty} \mathbb{E}\big[Z_{t\wedge\tau_m}^{(p-1)(1-\eta)}\big] &\le 1 + \lim_{m\to\infty} \mathbb{E}[Z_{t\wedge\tau_m}]\\&\le 1 + \theta t + \theta^2 \int_0^t s e^{\theta(t-s)}\, ds\end{aligned}$$

so that $\int_0^t \mathbb{E}\big[Z_s^{(p-1)(1-\eta)}\big]\, ds < \infty$. This together with (3.5) implies that the righthand side of (3.4) is finite.

Now that the finiteness of (3.3) is verified, (3.2) and monotone convergence prove the convergence

$$II_t^\epsilon \overset{\epsilon\to 0}{\longrightarrow} \int_0^t\int_0^\infty \Big((Z_{s-}+Z_{s-}^\beta\, x)^{1-\eta}-Z_{s-}^{1-\eta}\Big)\,(\mathcal{N}-\mathcal{N}')(ds,dx)$$

in L^p. Hence, there is a subsequence ϵ_k along which almost surely

$$\lim_{\epsilon_k\to 0} II_t^{\epsilon_k} = \int_0^t\int_0^\infty \Big((Z_{s-}+Z_{s-}^\beta\, x)^{1-\eta}-Z_{s-}^{1-\eta}\Big)\,(\mathcal{N}-\mathcal{N}')(ds,dx).$$

Finally, along ϵ_k all summands $I_t^{\epsilon_k}$, $II_t^{\epsilon_k}$, $III_t^{\epsilon_k}$ converge almost surely so that we proved the semimartingale decomposition

$$\begin{aligned} Z_t^{1-\eta} &= Z_0^{1-\eta} + (1-\eta)\int_0^t \Big(f(Z_s)-\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\Big)\,\mathbb{1}_{\{Z_s>0\}}\,ds \\ &\quad + \int_0^t\int_0^\infty \Big((Z_{s-}+Z_{s-}^\beta\, x)^{1-\eta}-Z_{s-}^{1-\eta}\Big)\,(\mathcal{N}-\mathcal{N}')(ds,dx), \end{aligned}$$

so that, replacing Z by $V^{1/(1-\eta)}$, the claim follows. □

Here is the reverse power transformation with a small but crucial difference in the drift.

Lemma 3.2. *Suppose that α,β and f are as in Theorem 1.5 and suppose that there exists a non-negative (strong) solution $(V_t)_{t\geq 0}$ to*

$$\begin{aligned} V_t &= V_0 + (1-\eta)\int_0^t \Big(f(V_s^{1/(1-\eta)})-\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\Big)\,ds \\ &\quad + \int_0^t\int_0^\infty \Bigg(\Big(V_{s-}^{\frac{1}{1-\eta}}+V_{s-}^{\frac{\beta}{1-\eta}}\,x\Big)^{1-\eta}-V_{s-}\Bigg)\,(\mathcal{N}-\mathcal{N}')(ds,dx), \quad t\geq 0, \end{aligned} \tag{3.6}$$

started at $V_0>0$. Then $Z:=V^{\frac{1}{1-\eta}}$ is a non-negative (strong) solution of the SDE (1.4) with initial condition $V_0^{\frac{1}{1-\eta}}$.

Proof. Applying the Meyer-Itô formula (see Theorem 51 of Protter [22]) with the convex function $F(v)=z^{1/(1-\eta)}$, we obtain

$$\begin{aligned} Z_t = V_t^{\frac{1}{1-\eta}} &= V_0^{\frac{1}{1-\eta}} + \int_0^t V_s^{\eta/(1-\eta)} f(V_s^{1/(1-\eta)})\,ds \\ &\quad + \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\int_0^t V_s^{\frac{\eta}{1-\eta}}\,ds + \int_0^t\int_0^\infty V_{s-}^{\frac{\beta}{1-\eta}}x\,(\mathcal{N}-\mathcal{N}')(ds,dx) \\ &\quad + \int_0^t\int_0^\infty \Bigg[V_s^{\frac{\beta}{1-\eta}}x - \frac{1}{1-\eta}V_s^{\frac{\eta}{1-\eta}}\Bigg(\Big(V_s^{\frac{1}{1-\eta}}+V_s^{\frac{\beta}{1-\eta}}\,x\Big)^{1-\eta}-V_s\Bigg)\Bigg]\,\mathcal{N}'(ds,dx) \\ &= Z_0 + \int_0^t Z_s^\eta f(Z_s)\,ds - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\int_0^t Z_s^\eta\,ds + \int_0^t\int_0^\infty Z_{s-}^\beta x\,(\mathcal{N}-\mathcal{N}')(ds,dx) \\ &\quad - \frac{1}{1-\eta}\int_0^t Z_s^\eta \int_0^\infty \Big[\big(Z_s+Z_s^\beta\, x\big)^{1-\eta}-Z_s^{1-\eta}-(1-\eta)Z_s^{\beta-\eta}x\Big]\,c_\alpha x^{-1-\alpha}\,dx\,ds. \end{aligned}$$

With the same integral identity used, in the proof of Lemma 3.1 for analyzing III^ϵ, we get that the inner integral in the last term on the righthand side equals to

$$\int_0^\infty G(Z_s, x, 0)\, c_\alpha\, x^{-1-\alpha}\, dx = -(1-\eta)\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}.$$

Substituting this into (3), we finally obtain

$$\begin{aligned} Z_t &= Z_0 + \int_0^t Z_s^\eta f(Z_s)\, ds + \int_0^t \int_0^\infty Z_{s-}^\beta x\, (\mathcal{N} - \mathcal{N}')(ds, dx) \\ &= Z_0 + \int_0^t Z_s^\eta f(Z_s)\, ds + \int_0^t Z_{s-}^\beta\, dL_s, \end{aligned}$$

so that the claim is proved. □

Before coming to the consequences of the power transformation we need existence and uniqueness for solutions of the jump type SDE (3.6).

Lemma 3.3. *Suppose that α, β, η and f are as in Theorem 1.5. If $f(0) \geq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ and $V_0 \geq 0$, then there is a unique non-negative strong solution V of the SDE (3.6). Moreover, if $f(0) > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, then $V \in \mathcal{S}$.*

Proof. Let $f(0) \geq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. To ease notation, let us define

$$\begin{aligned} g(v, z) &= v(1 + v^{-1/\alpha} z)^{1-\eta} - v \\ c(v) &= (1-\eta)\left(f(v^{1/(1-\eta)}) - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right) \end{aligned}$$

so that the SDE (3.6) can be written as

$$V_t = V_0 + \int_0^t c(V_s)\, ds + \int_0^t \int_0^\infty g(V_{s-}, x)(\mathcal{N} - \mathcal{N}')(ds, dx).$$

In the following we aim at applying the techniques developed in Li and Mytnik [20] though their Theorem 2.2 cannot be applied directly. First, for the drift we use the assumption on f to see that

$$|c(v_1) - c(v_2)| = (1-\eta)|f(v_1^{1/(1-\eta)}) - f(v_2^{1/(1-\eta)})| \leq K|v_1^{1/(1-\eta)} - v_2^{1/(1-\eta)}|$$

for some $K \in [0, \infty)$. Since $x \mapsto x^{1/(1-\eta)}$ is locally Lipschitz for $\eta < 1$ we find that the drift condition (2.b) of Li and Mytnik [20] is satisfied.

To deal with the jumps, let us start with some estimates for g. Taking the derivative with respect to v yields

$$\begin{aligned} \frac{\partial g}{\partial v}(v, z) &= -\frac{1-\eta}{\alpha}\frac{z}{v^{1/\alpha}}\left(1 + \frac{z}{v^{1/\alpha}}\right)^{-\eta} + \left(1 + \frac{z}{v^{1/\alpha}}\right)^{1-\eta} - 1 \\ &= -(1-\beta)x\,(1+x)^{-\eta} + (1+x)^{1-\eta} - 1 \\ &=: h(x), \end{aligned}$$

where $x = \frac{z}{v^{1/\alpha}}$. It follows directly that $h(0) = 0$ and furthermore

$$\begin{aligned} h'(x) &= (1-\beta)\eta x\,(1+x)^{-\eta-1} - (1-\beta)\,(1+x)^{-\eta} + (1-\eta)(1+x)^{-\eta} \\ &= (1-\beta)\,(1+x)^{-\eta-1}\left((\eta x - (1+x) + \frac{1-\eta}{1-\beta}(1+x)\right) \\ &= (1-\beta)\,(1+x)^{-\eta-1}\,(\alpha\beta x + \alpha - 1) \\ &> 0 \end{aligned}$$

for all $x > 0$ whereas the last equality follows as we recall the definition of $\eta = 1 - \alpha(1-\beta)$, and the last inequality follows from the assumption that $\alpha > 1$. This implies that $h(x)$ is positive for all $x > 0$, and hence $g(v, z)$ is increasing in v for all z. With this preparation we can find a modulus of continuity for g. Using the bound

$$(1+x)^{1-\eta} \le 1 + (1-\eta)x,\ x \ge 0, \tag{3.7}$$

shows that

$$0 \le h(x) \le 1 + (1-\eta)x - 1 = (1-\eta)x.$$

Now assume without loss of generality that $v_1 \le v_2$. To estimate $|g(v_2, z) - g(v_1, z)|$ we consider two cases:

Case 1. We first assume that $|v_2 - v_1| \le \frac{1}{2}v_2$. The previous calculations and the mean-value theorem yield (recall that $x = \frac{z}{v_1^{1/\alpha}}$),

$$|g(v_2, z) - g(v_1, z)| \le (1-\eta)x(v_2 - v_1) \le (1-\eta)\frac{z}{v_1^{1/\alpha}}(v_2 - v_1),$$

which combined with

$$v_1 \ge \frac{1}{2}v_2 \ge v_2 - v_1$$

gives the estimate

$$|g(v_2, z) - g(v_1, z)| \le (1-\eta)z(v_2 - v_1)^{1-1/\alpha}.$$

Case 2. Next we assume $|v_2 - v_1| \ge \frac{1}{2}v_2$. In this case we will use the following (crude) bound (recall that again $v_2 \ge v_1$ and $g(v, z)$ is increasing in v):

$$\begin{aligned} |g(v_2, z) - g(v_1, z)| \le g(v_2, z) &\le v_2(1 + (1-\eta)v_2^{-1/\alpha}z - 1) \\ &= (1-\eta)v_2^{1-1/\alpha}z \\ &\le c|v_2 - v_1|^{1-1/\alpha}z, \end{aligned}$$

where the second inequality follows from (3.7) and the last inequality follows from the assumption of Case 2.

In total we obtain the following uniform modulus of continuity for g:

$$|g(v_2, z) - g(v_1, z)| \le c|v_2 - v_1|^{1-1/\alpha}z. \tag{3.8}$$

We are now in a position to prove existence for all $t \ge 0$ and pathwise uniqueness for (3.6).

Pathwise Uniqueness: The claim follows from Proposition 3.1 of Li and Mytnik [20] for which conditions i)-iii) are trivially matched. As for condition (iv) from that proposition one can follow almost line by line the argument in the proof of Theorem 4.2 of [20], where the case of the critical Hölder exponent $p = 1 - 1/\alpha$ for the noise coefficient has been treated for a similar equation. Since in our case the Hölder exponent is also equal to $1 - 1/\alpha$ (see (3.8)), the argument goes through here as well. We leave the details to the reader.

Strong Existence: With the pathwise uniqueness in hands, strong solutions can now be constructed as in Section 5 of Fu and Li [13]; we only sketch the arguments. First, one has to consider the truncated equations

$$V_t = V_0 + \int_0^t c(V_s)\, ds + \int_0^t \int_\epsilon^m g(V_{s-}, x) \wedge m\, (\mathcal{N} - \mathcal{N}')(ds, dx),$$

which have solutions $V^{\epsilon,m}$ due to Theorem 4.4 of Fu and Li [13]. It follows readily from Aldous' criterion that the sequence $V^{\epsilon,m}$ is tight for any m fixed. Using the generators one can then verify weak convergence to a solution of

$$V_t = V_0 + \int_0^t c(V_s)\, ds + \int_0^t \int_0^m g(V_{s-}, x) \wedge m\, (\mathcal{N} - \mathcal{N}')(ds, dx).$$

The pathwise uniqueness proof given above applies equally for this truncated version so that any subsequences $V^{\epsilon_k,m}$ converge to the unique strong solution V^m. The pathwise uniqueness then allows to get rid of the truncation m as in the proof of Proposition 2.4 of Fu and Li [13].

Type $\mathcal{S}$: Let $f(0) > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. Suppose V is the unique strong solution of the SDE (3.6) constructed above. Then, by Lemma 3.2, $Z := V^{1/(1-\eta)}$ solves (1.1) and $Z^{1-\eta}$ solves the SDE (3.1). Since $V = (V^{1/(1-\eta)})^{1-\eta}$ this shows that V satisfies (3.1) and (3.6) so that equalizing both equations yields almost surely

$$\begin{aligned} &V_0 + (1-\eta)\left(\int_0^t \left(f(V_s^{1/(1-\eta)}) - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right) \mathbb{1}_{\{V_s \neq 0\}}\, ds\right) \\ &\quad + \int_0^t \int_0^\infty \left(\left(V_{s-}^{\frac{1}{1-\eta}} + V_{s-}^{\frac{\beta}{1-\eta}}\, x\right)^{1-\eta} - V_{s-}\right) (\mathcal{N} - \mathcal{N}')(ds, dx) \\ &= V_0 + (1-\eta) \int_0^t \left(f(V_s^{1/(1-\eta)}) - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right) ds \\ &\quad + \int_0^t \int_0^\infty \left(\left(V_{s-}^{\frac{1}{1-\eta}} + V_{s-}^{\frac{\beta}{1-\eta}}\, x\right)^{1-\eta} - V_{s-}\right) (\mathcal{N} - \mathcal{N}')(ds, dx). \end{aligned}$$

Hence, since $f(0) > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, we get $\int_0^t \mathbb{1}_{\{V_s \neq 0\}}\, ds = t$ almost surely which proves that V is of type $\mathcal{S}$. □

Lemma 3.4. *Suppose that α, β, η and f are as in Theorem 1.5.*

A) *If $f(0) > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ and $V_0 \geq 0$, then pathwise uniqueness holds for solutions of the SDE (3.1) in $\mathcal{S}$. Moreover, a strong solution in $\mathcal{S}$ to* (3.1) *exists.*

B) *If* $f(0) \le \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ *and* $V_0 \ge 0$*, then there exists a unique strong non-negative solution to* (3.1)*. If the solution hits zero, then it stays at zero forever.*

Proof. A) By Lemma 3.3, there is a unique non-negative strong solution V of the SDE (3.6) and $V \in \mathcal{S}$. Since $V \in \mathcal{S}$, it also solves (3.1), and this implies existence of the strong solution to (3.1). As for the pathwise uniqueness, if V^1, V^2 are two solutions to (3.1) in $\mathcal{S}$ starting at the same initial value V_0, then they are also two solutions to (3.6). Therefore they are equal by Lemma 3.3, and hence the pathwise uniqueness holds for solutions of (3.1) in $\mathcal{S}$.

B) We already saw that $T_0 < \infty$ with positive probability. Since pathwise uniqueness holds up to T_0, we only need to show that under the assumption of B) solutions are trapped at zero. First, we will treat the case of $f(0) < \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$.
We begin with proving that if $V_0 = 0$ almost surely, then $V_t = 0$ almost surely for any $t \ge 0$. By our assumption $f(0) < \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, and the continuity of f, there is $\delta > 0$ such that

$$f(x^{1/(1-\eta)}) - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < 0, \quad \forall\, x \in [0, \delta]. \tag{3.9}$$

If T_δ denotes the first hitting time of $[\delta, \infty)$, then, by Fatou's lemma,

$$0 \le \mathbb{E}[V_{T_\delta} \mathbb{1}_{\{T_\delta < \infty\}}] \le \liminf_{t\to\infty} \mathbb{E}[V_{t\wedge T_\delta} \mathbb{1}_{\{T_\delta<\infty\}}] \le \liminf_{t\to\infty} \mathbb{E}[V_{t\wedge T_\delta}]. \tag{3.10}$$

Next, we show that the righthand side equals zero since the expectation is equal to zero for any $t > 0$. But this follows directly from the defining equation:

$$\mathbb{E}[V_{t\wedge T_\delta}] = (1-\eta)\,\mathbb{E}_0\left[\int_0^{t\wedge T_\delta} \left(f((V_s)^{\frac{1}{1-\eta}}) - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right) \mathbb{1}_{\{V_s \neq 0\}}\, ds\right] \le 0,$$

due to (3.9) and the definition of T_δ. Plugging into the righthand side of (3.10) we see that the event $\{T_\delta < \infty\}$ must have zero probability. Since δ can be arbitrarily small this shows that solutions started at zero remain trapped at zero almost surely.

Now we need to reduce the case $V_0 > 0$ to the case $V_0 = 0$. Observe that the trivial function is a solution to (3.1) when we start from $V_0 = 0$ so it makes sense to define

$$\bar{V}_t(\omega) = \begin{cases} 0 & : \omega \in \{T_0 = \infty\} \\ V_{T_0+t}(\omega) & : \omega \in \{T_0 < \infty\} \end{cases},$$

i.e. for paths of V not hitting zero we set $\bar{V}$ equal to zero. In order to argue that $\bar{V}$ is a solution to (3.1), which by definition has initial condition $\bar{V}_0 = 0$ almost surely, we need to define a suitable driving Poisson point process. For a Poisson point process $\mathcal{M}$ independent of $\mathcal{N}$ but with same intensity, we set

$$\bar{\mathcal{N}}(ds, dx)(\omega) = \begin{cases} \mathcal{M}(ds, dx)(\omega) & : \omega \in \{T_0 = \infty\} \\ \mathcal{N}(T_0 + ds, dx)(\omega) & : \omega \in \{T_0 < \infty\} \end{cases}.$$

Then $\bar{\mathcal{N}}$ is a Poisson point process with intensity $ds \otimes c_\alpha x^{-1-\alpha}\, dx$ with respect to which $\bar{V}$ is a solution to (3.1) started at zero. Note that on $\{T_0 = \infty\}$ the solution is constant zero so there is no measurability problem involved. By the above reasoning $\bar{V}$ has to be equal to zero almost surely and, hence, V is trapped at zero.

Let us finally settle the case $f(0) = \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. Here, one can readily get that any non-negative solution to (3.1) is, in fact, a solution to (3.6) and vice versa. Moreover, there exists a solution V to (3.1) (and also to (3.6)) such that, on the event $\{T_0 < \infty\}$, $V_{T_0+t} \equiv 0$, for all $t \geq 0$. Then by the pathwise uniqueness result from Lemma 3.3 we get that this, in fact, is the unique strong non-negative solution to both (3.6) and (3.1). □

We are now prepared to prove Theorems 1.2 and 1.5.

Proof of Theorem 1.5. A) Let $(V_t)_{t\geq 0}$ be the unique in $\mathcal{S}$ strong solution to (3.1) constructed in Lemma 3.4 A). Since the solution is in $\mathcal{S}$ it also solves (3.6). Then Lemma 3.2 shows that $Z = V^{\frac{1}{1-\eta}}$ is a strong solution to the SDE (1.4) and remind again that it is of type $\mathcal{S}$. As for the pathwise uniqueness, suppose there are two solutions Z^1, Z^2 with $Z_0^1 = Z_0^2$ to (1.4) and that they spend a Lebesgue-negligible set of time at zero. Then the power transformation is truely reversible due to Lemma 3.1, so that the uniqueness follows from Lemma 3.4 A).

B) The proof goes along the similar lines as the proof of A). Let $(V_t)_{t\geq 0}$ be the unique non-negative strong solution to (3.1) constructed in Lemma 3.4 B). One can easily see that $Z = V^{\frac{1}{1-\eta}}$ is a strong solution to the SDE (1.4).

As for the pathwise uniqueness, suppose there are two strong non-negative solutions Z^1, Z^2 with $Z_0^1 = Z_0^2$ to (1.4). Use again Lemma 3.1 to see that the pathwise uniqueness follows from Lemma 3.4 B). □

Proof of Theorem 1.2. Theorem 1.2 is a straightforward consequence once we remark that, by Theorem 1.1 with $f \equiv \theta$, the solution Z of the SDE (1.1) always hits 0 in finite time if $\theta \leq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ since by Lemma 2.5 we have $\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < \Gamma(\alpha)$. □

4. Self-Similar Extensions

Lamperti's transformation for pssMps, which we recalled in Section 2, can not be used directly to characterize pssMps after hitting zero (or started from zero) since the infinite time-horizon $(0,\infty)$ for ξ is compressed via the time-change to the possibly finite time-horizon $(0, T_0)$ so that the entire information on ξ is already used until T_0. This drawback was resolved in recent years.

If ξ drifts to $-\infty$, Rivero [25] and Fitzsimmons [11] independently proved that a pssMp of index γ has a recurrent self-similar Markovian extension of index γ after T_0 with non-negative sample paths that leave zero continuously if and only if the following Cramér-type condition holds for the corresponding Lévy process ξ:

$$\text{There is } 0 < a < \frac{1}{\gamma} \text{ such that } \mathbb{E}[e^{a\xi_1}] = 1. \tag{4.1}$$

Due to Corollary 2.4 ii), Condition (4.1) is equivalent to $\theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ which we found more directly to be the necessary and sufficient condition for the existence of non-trivial solutions to the SDE (1.1).

To further explore the connection between the present work and the results of Rivero and Fitzsimmons recall from Lemma 2.1 that solutions to the SDE (1.1) define a pssMp $(\mathbb{P}^x)_{x>0}$. If we define furthermore $(\bar{\mathbb{P}}^x)_{x\geq 0}$ via the solutions $Z^x \in \mathcal{S}$ to the SDE (1.1) started at $x \geq 0$, then we can easily deduce the following consequence from the pathwise uniqueness:

Corollary 4.1. *Let $\beta \in [1-1/\alpha, 1)$ and suppose $\Gamma(\alpha) > \theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. Then $(\bar{\mathbb{P}}^x)_{x\geq 0}$ is the unique extension of $(\mathbb{P}^x)_{x>0}$ that leaves zero continuously.*

Proof. It follows directly from the definition of $(\bar{\mathbb{P}}^x)_{x\geq 0}$ that it is an extension of $(\mathbb{P}^x)_{x>0}$ so that it suffices to prove the self-similarity for our solutions $Z^x \in \mathcal{S}$ to the SDE (1.1). Since, by construction, those are obtained by taking the power $1-\eta$ of solutions to the SDE (3.6) it suffices to show that solutions to the SDE (3.6) are self-similar of index 1. Setting $V_t^c := cV_{tc^{-1}}$ and plugging into the defining equation yields

$$\begin{aligned}
V_t^c = cV_{tc^{-1}} &= cV_0 + (1-\eta)\left(\theta - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right)t \\
&\quad + \int_0^{tc^{-1}}\int_0^\infty \left(\left(\left(cV_{s-}\right)^{\frac{1}{1-\eta}} + \left(cV_{s-}\right)^{\frac{\beta}{1-\eta}} c^{\frac{1}{\alpha}} x\right)^{1-\eta} - cV_{s-}\right)(\mathcal{N}-\mathcal{N}')(ds,dx) \\
&= V_0^c + (1-\eta)\left(\theta - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right)t \\
&\quad + \int_0^t\int_0^\infty \left(\left(\left(V_{s-}^c\right)^{\frac{1}{1-\eta}} + \left(V_{s-}^c\right)^{\frac{\beta}{1-\eta}} x\right)^{1-\eta} - V_{s-}^c\right)(\mathcal{N}_{(c)}-\mathcal{N}'_{(c)})(ds,dx),
\end{aligned}$$

where $\mathcal{N}_{(c)}$ and $\mathcal{N}'_{(c)}$ are the image of $\mathcal{N}$, respectively $\mathcal{N}'$, under the map $(s,x) \mapsto (cs, c^{1/\alpha}x)$. Since $\mathcal{N}'_{(c)} = \mathcal{N}'$, $\mathcal{N}_{(c)}$ has the same law as $\mathcal{N}$, and we see that both $(V_t)_{t\geq 0}$ and $(cV_{tc^{-1}})_{t\geq 0}$ are solutions to the same well-posed SDE so that they coincide in law. Solutions trivially leave zero continuously since the integrand is null at zero. $\square$

If ξ does not drift to $-\infty$, by (2.2) almost surely the sample paths of the corresponding pssMp do not hit zero. The main question becomes whether the Markov family $(\mathbb{P}^x)_{x>0}$ can be extended continuously to $\mathbb{P}^0$. Caballero and Chaumont [8] and later Chaumont et al. [9] proved that $\mathbb{P}^x$ converges weakly to a non-trivial limit law as x tends to zero, i.e. it is a Feller process on $[0,\infty)$ and not on $(0,\infty)$ only, if and only if the overshoot process

$$\xi_{T_x} - x,\ x \geq 0, \quad \text{with} \quad T_x := \inf\{t \geq 0 : \xi_t \geq x\},$$

converges, as $x \to \infty$, weakly towards the law of a finite random variable. A simpler construction of $\mathbb{P}^0$ has been given in Bertoin and Savov [6] via Lévy processes indexed by the real line.

In the case of the pssMps $(\mathbb{P}^x)_{x>0}$ corresponding to the SDE (1.1), the Feller property on $[0,\infty)$ is again a direct consequence of the uniqueness of Lemma 3.4.

Corollary 4.2. *Let $\beta \in [1-1/\alpha, 1)$ and suppose that $\theta > \Gamma(\alpha)$. Then $(\bar{\mathbb{P}}^x)_{x\geq 0}$ is weakly continuous in the initial condition.*

Proof. This follows directly from the uniqueness of Lemma 3.4 combined with [15, Theorem IX.4.8] taking into account Lemma 2.5. □

Our direct expression of the self-similar extension at zero is possible since the pssMp is given by a stochastic differential equation. In Döring and Barczy [2] this approach is extended by first reformulating Lamperti's transformation via jump type SDEs and then proceeding accordingly.

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